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A new integral equation method for direct electromagnetic scattering in homogeneous media and its numerical confirmation

J. DeSANTO*† and A. YUFFA‡

†Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, Colorado 80401, USA
‡Department of Physics, Colorado School of Mines, Golden, Colorado 80401, USA

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In this paper, we derive a new integral equation method for direct electromagnetic scattering in homogeneous media and present a numerical confirmation of the new method via a computer simulation. The new integral equation method is based on a paper written by DeSanto [1], originally for scattering from an infinite rough surface separating homogeneous dielectric half-spaces. Here, it is applied to a bounded scatterer, which can be an ohmic conductor or a dielectric, with some simplification of the continuity conditions for the fields. The new integral equation method is developed by choosing the electric field and its normal derivative as boundary unknowns, which are not the usual boundary unknowns. The new integral equation method may provide significant computational advantages over the standard Stratton–Chu method [2] because it leads to a 50% sparse, rather than 100% dense, impedance (collocation) matrix. Our theoretical development of the new integral equation method is exact.

1. Introduction

Three-dimensional space is divided into two regions. Each region has a constant permittivity and permeability; in general, these values may be complex. Region two, the scatterer, is bounded by a smooth surface given by

$$S^{\pm}(x, y) = \begin{cases} S^{+}(x, y) & \text{if} \quad z \geq 0 \\ S^{-}(x, y) & \text{if} \quad z \leq 0, \end{cases} \tag{1}$$

where the domain of $S^{\pm}(x, y)$ is a disk with radius r_o in the $z = 0$ plane. The non-unit normal to the surface defined by (1) is given by

$$\mathbf{n}^{\pm}(x, y) = \begin{cases} \mathbf{n}^{+}(x, y) & \text{if} \quad z \geq 0 \\ \mathbf{n}^{-}(x, y) & \text{if} \quad z \leq 0, \end{cases} \tag{2}$$

where $n_i^{+}(x, y) = \partial_i(z - S^{+}(x, y))$ and $n_i^{-}(x, y) = \partial_i(S^{-}(x, y) - z)$. Throughout this paper, we will use index notation and the SI unit system, and we will assume that all fields are harmonic in time with a $\exp(-\mathrm{i}\omega t)$ time factor, where ω is the angular frequency. By index notation,

*Corresponding author. E-mail: jdesanto@mines.edu

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we mean that every index letter appearing as a repeated subscript in one term indicates a summation from one to three. If the subscript also contains the letter t (transverse), then the summation is from one to two. Every component of the electric field in region two and every component of the scattered electric field in region one satisfies the Helmholtz equation

$$\left(\partial'_j \partial'_j + k_R^2\right) E_i^{(R)}(\mathbf{x}') = 0, \tag{3}$$

where $\mathbf{x}' = (x', y', z')$, R is the region number ($R = 1, 2$), and k is a complex wavenumber. The imaginary part of k is chosen such that $\mathrm{Im}(k) \geq 0$. In order to obtain an integral representation of the fields, we define two functions. The first function $G^{(\mathrm{R})}(\mathbf{x}, \mathbf{x}')$ is termed a free-space Green's function. It satisfies

$$\left(\partial'_j \partial'_j + k_R^2\right) G^{(R)}(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x}' - \mathbf{x}), \tag{4}$$

where $\delta(\mathbf{x}' - \mathbf{x})$ is the Dirac delta function. The free-space Green's function is known explicitly to be

$$G^{(R)}(\mathbf{x}, \mathbf{x}') = \frac{\exp\left(\mathrm{i}k_R|\mathbf{x} - \mathbf{x}'|\right)}{4\pi|\mathbf{x} - \mathbf{x}'|}, \tag{5}$$

where $|\mathbf{x} - \mathbf{x}'|$ denotes the magnitude of $\mathbf{x} - \mathbf{x}'$. The second function, $\Theta^{(R)}(\mathbf{x}')$, is termed the characteristic function and is defined by

$$\Theta^{(\mathrm{R})}(\mathbf{x}') = \begin{cases} 1 & \text{if} \quad \mathbf{x}' \in \text{region } R \\ 0 & \text{if} \quad \mathbf{x}' \notin \text{region } R. \end{cases} \tag{6}$$

The characteristic functions for regions two and one are explicitly given by

$$\Theta^{(2)}(\mathbf{x}') = \Theta(r_o - r')\Theta(S^+(x', y') - z')\Theta(z' - S^-(x', y')), \tag{7}$$

$$\Theta^{(1)}(\mathbf{x}') = 1 - \Theta^{(2)}(\mathbf{x}'), \tag{8}$$

where $r' = \sqrt{x'^2 + y'^2}$. To obtain an integral representation of the fields, we multiply (3) by $G^{(R)}(\mathbf{x}, \mathbf{x}')$ and (4) by $E_i^{(R)}(\mathbf{x}')$, then take the difference between the two equations and multiply it by the characteristic function for region R. Then, we integrate the result over all space with respect to the primed coordinates. This yields

$$E_i^{(R)}(\mathbf{x})\Theta^{(\mathrm{R})}(\mathbf{x}) = -\int \left[G^{(R)}(\mathbf{x}, \mathbf{x}')\partial'_j E_i^{(R)}(\mathbf{x}') - E_i^{(R)}(\mathbf{x}')\partial'_j G^{(R)}(\mathbf{x}, \mathbf{x}')\right]\partial'_j \Theta^{(\mathrm{R})}(\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{9}$$

where the terms integrated by parts vanish because the characteristic function vanishes outside region R. The total field in region one is composed of the incident field and the scattered field. The scattered field satisfies the Helmholtz equation and the Silver–Müller radiation condition [3]; thus, any integral over a surface at infinity will vanish. The total field in region one can be written as

$$\mathbf{E}^{(1)} = \mathbf{E}^{(s)} + \mathbf{E}^{(\mathrm{inc})}, \tag{10}$$

where $\mathbf{E}^{(s)}$ is the scattered electric field and $\mathbf{E}^{(\mathrm{inc})}$ is the incident electric field. Writing (9) for $\mathbf{E}^{(s)}$ (with $R = 1$) and using (10), and then integrating the result with respect to z' yields

$$\begin{aligned} &\left(E_i^{(\mathrm{inc})}(\mathbf{x}) - E_i^{(1)}(\mathbf{x})\right)\Theta^{(1)}(\mathbf{x}) + W_i^{(\mathrm{inc})}(\mathbf{x}) \\ &\quad = \int \left[G^{(1)}(\mathbf{x}, \mathbf{x}'_{S^+})n_j^+ \partial'_j E_i^{(1)} - E_i^{(1)} n_j^+ \partial'_j G^{(1)}(\mathbf{x}, \mathbf{x}'_{S^+})\right]\Theta(\varrho)\,\mathrm{d}\mathbf{x}'_{it} \\ &\quad + \int \left[G^{(1)}(\mathbf{x}, \mathbf{x}'_{S^-})n_j^- \partial'_j E_i^{(1)} - E_i^{(s)} n_j^- \partial'_j G^{(1)}(\mathbf{x}, \mathbf{x}'_{S^-})\right]\Theta(\varrho)\,\mathrm{d}\mathbf{x}'_{it}, \end{aligned} \tag{11}$$

where $\mathbf{x}'_{S^\pm} = (x', y', S^\pm(x', y'))$, $\varrho = r_o - r'$, and $W_i^{(\mathrm{inc})}(\mathbf{x})$ is given by

$$W_i^{(\mathrm{inc})}(\mathbf{x}) = \int \left[G^{(1)}(\mathbf{x}, \mathbf{x}'_{S^+}) n_j^+ \partial'_j E_i^{(\mathrm{inc})} - E_i^{(\mathrm{inc})} n_j^+ \partial'_j G^{(1)}(\mathbf{x}, \mathbf{x}'_{S^+})\right] \Theta(\varrho)\, \mathrm{d}\mathbf{x}'_{it} + \int \left[G^{(1)}(\mathbf{x}, \mathbf{x}'_{S^-}) n_j^- \partial'_j E_i^{(\mathrm{inc})} - E_i^{(\mathrm{inc})} n_j^- \partial'_j G^{(1)}(\mathbf{x}, \mathbf{x}'_{S^-})\right] \Theta(\varrho)\, \mathrm{d}\mathbf{x}'_{it}. \quad (12)$$

Writing (9) for $\mathbf{E}^{(2)}$ (with $R = 2$) and integrating the result with respect to z' yields

$$E_i^{(2)}(\mathbf{x})\Theta^{(2)}(\mathbf{x}) = \int \left[G^{(2)}(\mathbf{x}, \mathbf{x}'_{S^+}) n_j^+ \partial'_j E_i^{(2)} - E_i^{(2)} n_j^+ \partial'_j G^{(2)}(\mathbf{x}, \mathbf{x}'_{S^+})\right] \Theta(\varrho)\, \mathrm{d}\mathbf{x}'_{it} + \int \left[G^{(2)}(\mathbf{x}, \mathbf{x}'_{S^-}) n_j^- \partial'_j E_i^{(2)} - E_i^{(2)} n_j^- \partial'_j G^{(2)}(\mathbf{x}, \mathbf{x}'_{S^-})\right] \Theta(\varrho)\, \mathrm{d}\mathbf{x}'_{it}. \quad (13)$$

The integral representation of the fields given by (11) and (13) may be used to compute the electric fields in regions one and two, respectively, once the electric field and its normal derivative are known on the surface. In order to form six scalar coupled linear integral equations from (11) and (13) that can be solved numerically once appropriate boundary conditions are chosen, we let the field point $\mathbf{x}$ approach the surface. It is well known that the Green's function and its normal derivative are singular when the field point coincides with the source point, $\mathbf{x}'_{S^\pm}$. The singularity of the Green's function can be integrated to obtain 0, and the singularity of the normal derivative of the Green's function can be integrated to obtain $\pm 1/2$. The value $+1/2$ is obtained if we approach the surface from the region into which the normal points, and $-1/2$ is obtained if we approach the surface from the region opposite that into which the normal points. In the limit as $\mathbf{x}$ approaches the surface from region one, (11) yields

$$\frac{1}{2}\left(E_i^{(\mathrm{inc})}(\mathbf{x}_{S^\pm}) - E_i^{(1)}(\mathbf{x}_{S^\pm})\right) = ⨍ \left[G^{(1)}(\mathbf{x}_{S^\pm}, \mathbf{x}'_{S^+}) n_j^+ \partial'_j E_i^{(1)} - E_i^{(1)} n_j^+ \partial'_j G^{(1)}(\mathbf{x}_{S^\pm}, \mathbf{x}'_{S^+})\right] \Theta(\varrho)\, \mathrm{d}\mathbf{x}'_{it} + ⨍ \left[G^{(1)}(\mathbf{x}_{S^\pm}, \mathbf{x}'_{S^-}) n_j^- \partial'_j E_i^{(1)} - E_i^{(1)} n_j^- \partial'_j G^{(1)}(\mathbf{x}_{S^\pm}, \mathbf{x}'_{S^-})\right] \Theta(\varrho)\, \mathrm{d}\mathbf{x}'_{it}, \quad (14)$$

where $⨍$ denotes the Cauchy principal value integral. In the derivation of (14), we have used the fact that

$$\lim_{\mathbf{x} \to \mathbf{x}_{S^\pm}} W_i^{(\mathrm{inc})}(\mathbf{x}) = -\frac{1}{2} E_i^{(\mathrm{inc})}(\mathbf{x}_{S^\pm})$$

if $\mathbf{E}_i^{(\mathrm{inc})}$ satisfies the Helmholtz equation. In the limit as $\mathbf{x}$ approaches the surface from region two, (13) yields

$$\frac{1}{2} E_i^{(2)}(\mathbf{x}_{S^\pm}) = ⨍ \left[G^{(2)}(\mathbf{x}_{S^\pm}, \mathbf{x}'_{S^+}) n_j^+ \partial'_j E_i^{(2)} - E_i^{(2)} n_j^+ \partial'_j G^{(2)}(\mathbf{x}_{S^\pm}, \mathbf{x}'_{S^+})\right] \Theta(\varrho)\, \mathrm{d}\mathbf{x}'_{it} + ⨍ \left[G^{(2)}(\mathbf{x}_{S^\pm}, \mathbf{x}'_{S^-}) n_j^- \partial'_j E_i^{(2)} - E_i^{(2)} n_j^- \partial'_j G^{(2)}(\mathbf{x}_{S^\pm}, \mathbf{x}'_{S^-})\right] \Theta(\varrho)\, \mathrm{d}\mathbf{x}'_{it}. \quad (15)$$

2. New boundary unknowns

We wish to choose the electric field and its normal derivative as the boundary unknowns. In order to choose these boundary unknowns, we must have continuity conditions for the electric field and its normal derivative. These continuity conditions were first derived by DeSanto

[1] for an uncharged dielectric scatterer. In appendix A, we generalize DeSanto's continuity conditions for uncharged ohmic conductors with finite conductivities. Substituting (45) into (15) and integrating any transverse terms by parts yields

$$\frac{1}{2}E_i^{(2)}(\mathbf{x}_{S^\pm}) = \fint \mu G^{(2)} n_j^+ \partial_j' E_i^{(1)} \Theta(\varrho)\, \mathrm{d}\mathbf{x}_{it}' + \fint \big[- E_i^{(2)} n_j^+ \partial_j' G^{(2)} + \big(\mu E_j^{(1)} - E_j^{(2)}\big) \times \partial_{it}' \{n_j^+ G^{(2)}\} + \big(E_{pt}^{(2)} - \mu E_{pt}^{(1)}\big) \partial_{pt}' \{n_i^+ G^{(2)}\}\big] \Theta(\varrho)\, \mathrm{d}\mathbf{x}_{it}' + \fint \mu G^{(2)} n_j^- \partial_j' E_i^{(1)} \Theta(\varrho)\, \mathrm{d}\mathbf{x}_{it}' + \fint \big[- E_i^{(2)} n_j^- \partial_j' G^{(2)} + \big(\mu E_j^{(1)} - E_j^{(2)}\big) \times \partial_{it}' \{n_j^- G^{(2)}\} + \big(E_{pt}^{(2)} - \mu E_{pt}^{(1)}\big) \partial_{pt}' \{n_i^- G^{(2)}\}\big] \Theta(\varrho)\, \mathrm{d}\mathbf{x}_{it}', \tag{16}$$

where $\mu = \mu_2/\mu_1$ and curly brackets, { }, indicate that the contained function has been set on the surface, $S^\pm$. It is understood that a function inside of curly brackets is to be differentiated as a function of only x' and y'. The terms in square brackets, [ ], in (16) can be simplified by using identities given by

$$\partial_{pt}' \{n_\ell^\pm G^{(2)}\} = \partial_{pt}' \big(n_\ell^\pm G^{(2)}\big) \mp n_{pt}^\pm \partial_3' \big(n_\ell^\pm G^{(2)}\big) = n_\ell^\pm \partial_{pt}' G^{(2)} + G^{(2)} \partial_{pt}' n_\ell^\pm \mp n_{pt}^\pm n_\ell^\pm \partial_3' G^{(2)}, \tag{17}$$

$$\partial_{it}' n_{pt}^\pm = \partial_{pt}' n_i^\pm \quad \text{if} \quad \partial_1' \partial_2' S^\pm = \partial_2' \partial_1' S^\pm, \tag{18}$$

and

$$E_j^{(\mathrm{R})} G^{(2)} \partial_{it}' n_j^\pm = E_{pt}^{(\mathrm{R})} G^{(2)} \partial_{pt}' n_i^\pm. \tag{19}$$

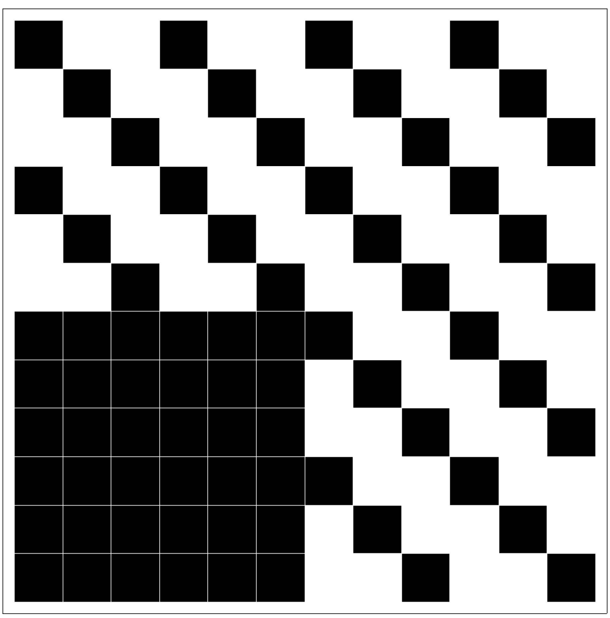

Figure 1. The impedance matrix for the new integral equation method is shown. The black squares represent non-zero block matrices and the white squares represent zero block matrices.

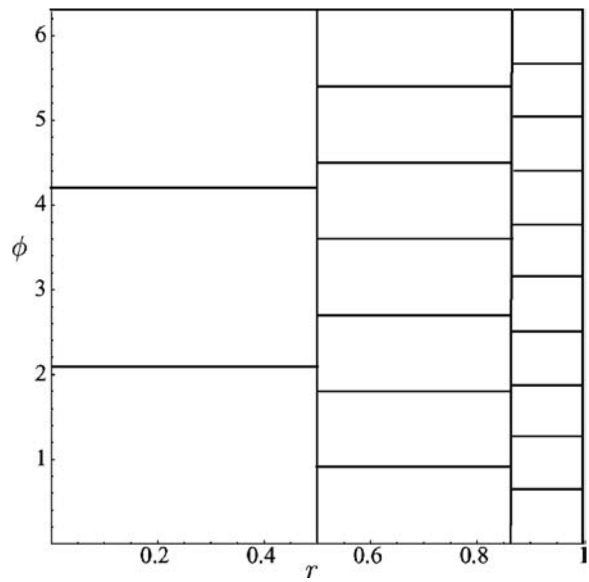

Figure 2. Each corner of each rectangle is defined by $r_i = r_o \sin(\frac{\pi i}{2n})$ for $0 \leq i \leq n$ and $\phi_j = \frac{2\pi j}{m_i}$ for $0 \leq j \leq m_\ell$, where $n = \lceil \frac{\pi r_o p}{2\lambda} \rceil$, $m_\ell = \lceil \frac{\pi(r_{\ell+1}+r_\ell)p}{\lambda} \rceil$ for $0 \leq \ell < n$, λ is the incident wavelength, and p is some integer. For the mesh shown above, $r_o = 1$, $\lambda = 2\pi$, $p = 10$, and $N_\square = 20$, where $N_\square$ denotes the number of rectangles.

Using identities (17)–(19) to simplify the terms in the square brackets in (16) yields

$$\begin{aligned} &- E_i^{(2)} n_j^{\pm} \partial'_j G^{(2)} + n_i^{\pm} E_j^{(2)} \partial'_j G^{(2)} - n_j^{\pm} E_j^{(2)} \partial'_i G^{(2)} \\ &+ \mu\big(n_j^{\pm} E_j^{(1)} \partial'_i G^{(2)} - n_i^{\pm} E_j^{(1)} \partial'_j G^{(2)}\big). \end{aligned} \tag{20}$$

Substituting (31) into (20) and simplifying the result yields

$$-E_i^{(1)} n_j^{\pm} \partial'_j G^{(2)} + (\mu - \epsilon^{-1}) n_j^{\pm} E_j^{(1)} \partial'_i G^{(2)} + (1 - \mu) n_i^{\pm} E_j^{(1)} \partial'_j G^{(2)}, \tag{21}$$

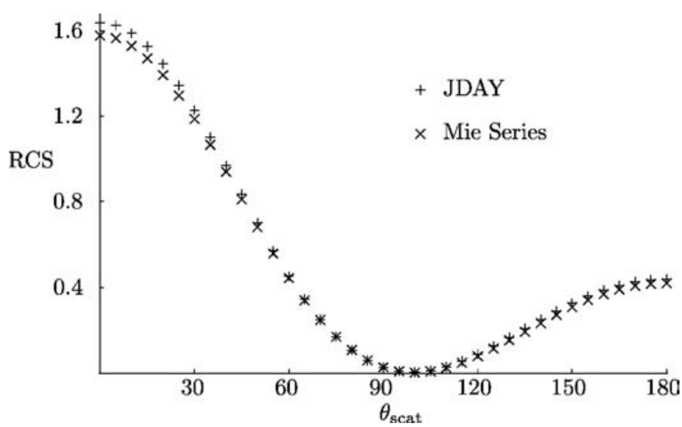

Figure 3. Case 1: $k_{\text{inc}} R = 1$, $\epsilon = 4$, $\mu = 1$, and the scatterer is a sphere. 1418 surface patches were used to obtain the JDAY solution.

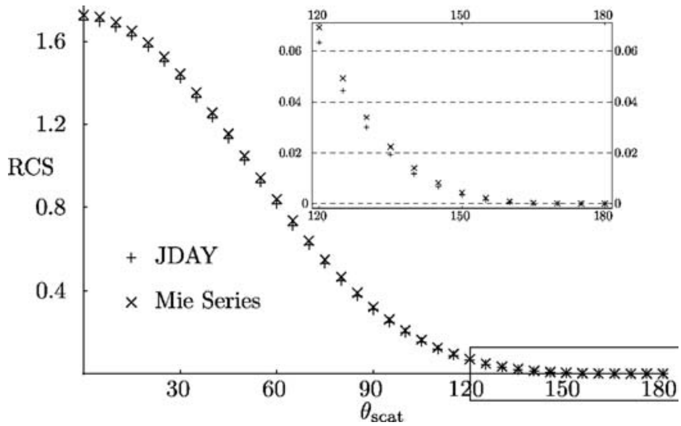

Figure 4. Case 2: $k_{\mathrm{inc}}R = 1$, $\epsilon = 2$, $\mu = 2$, and the scatterer is a sphere. 1418 surface patches were used to obtain the JDAY solution. Notice that both solutions satisfy Weston's theorem, see 'zoomed-in' plot.

where $\epsilon = \epsilon_2/\epsilon_1$. Finally, substituting (21) and (31) into (16) yields

$$\begin{aligned}\frac{1}{2}C_{ij}E_j^{(1)}(\mathbf{x}_{S^\pm}) = &\int \left[\mu G^{(2)} n_j^+ \partial_j' E_i^{(1)} - E_i^{(1)} n_j^+ \partial_j' G^{(2)}\right]\Theta(\varrho)\,\mathrm{d}\mathbf{x}_{it}' \\ &+ \int \left[(\mu - \epsilon^{-1}) n_j^+ E_j^{(1)} \partial_i' G^{(2)} + (1-\mu) n_i^+ E_j^{(1)} \partial_j' G^{(2)}\right]\Theta(\varrho)\,\mathrm{d}\mathbf{x}_{it}' \\ &+ \int \left[\mu G^{(2)} n_j^- \partial_j' E_i^{(1)} - E_i^{(1)} n_j^- \partial_j' G^{(2)}\right]\Theta(\varrho)\,\mathrm{d}\mathbf{x}_{it}' \\ &+ \int \left[(\mu - \epsilon^{-1}) n_j^- E_j^{(1)} \partial_i' G^{(2)} + (1-\mu) n_i^- E_j^{(1)} \partial_j' G^{(2)}\right]\Theta(\varrho)\,\mathrm{d}\mathbf{x}_{it}', \quad (22)\end{aligned}$$

where the arguments of the functions have been omitted to conserve space. Equations (14) and (22) form a system of six coupled linear integral equations, where the electric field, $\boldsymbol{E}^{(1)}$, and its normal derivative, $n_j^\pm \partial_j' E_i^{(1)}$, are the boundary unknowns. This system of coupled linear integral equations can be written in matrix form as

$$\mathbf{A}_{ij} f_j = g_i \quad 1 \le i, j \le 12, \quad (23)$$

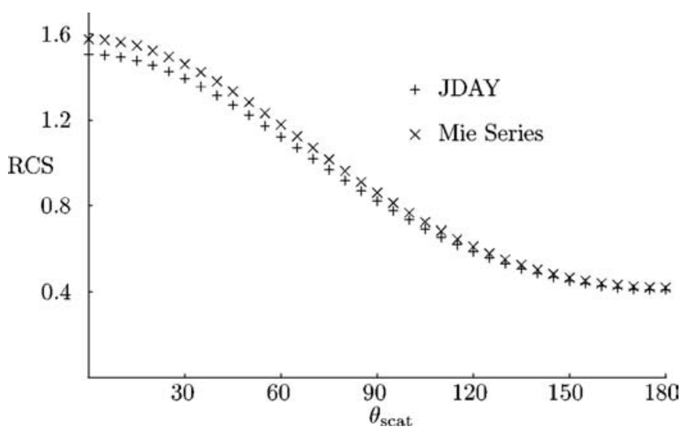

Figure 5. Case 3: $k_{\mathrm{inc}}R = 1$, $\epsilon = 1$, $\mu = 4$, and the scatterer is a sphere. 1418 surface patches were used to obtain the JDAY solution.

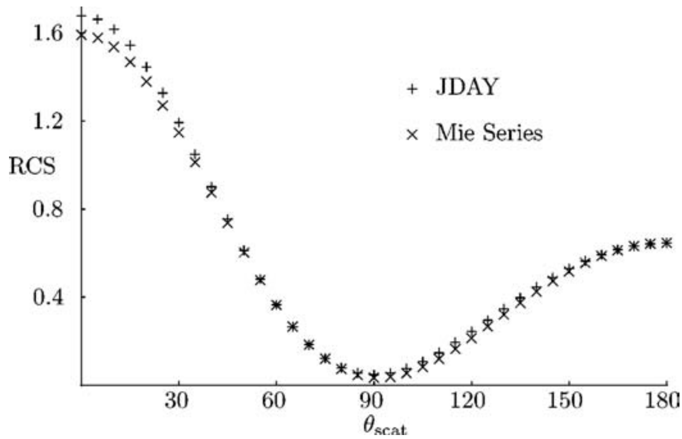

Figure 6. Case 4: $k_{inc}R = 1$, $\epsilon = 1 + 4i$, $\mu = 1$, and the scatterer is a sphere. 1418 surface patches were used to obtain the JDAY solution.

where $\mathbf{A}_{ij}$ is the impedance matrix, f_j is the unknown column vector, and g_i is the known column vector that contains the incident electric field. Moreover, the upper-half of the impedance matrix is generated from (14), and the lower-half of the impedance matrix is generated from (22). The upper-half of the unknown column vector, f_j, contains the electric field, and the lower-half contains the normal derivative of the electric field. The impedance matrix is shown in figure 1, where the white squares are the zero block matrices and the black squares are the non-zero block matrices. The upper-left-quadrant, the upper-right-quadrant, and the lower-right-quadrant of the impedance matrix are 'diagonal' and sparse in structure. The overall sparsity of the impedance matrix is 50%. The sparsity and the structure of the impedance matrix may provide a significant computational advantage over the standard Stratton–Chu integral equation method [2, 4, 5] because its corresponding impedance matrix is 100% dense.

3. Numerical results

In order to numerically confirm our new integral equation method, we have written a computer program called JDAY[1]. JDAY is based on the discrete collocation method with pulse basis functions. JDAY numerically solves our new integral equations, which are given by (14) and (22) for the new boundary unknowns, i.e. the electric field and its normal derivative. Equations (14) and (22) are transformed into a cylindrical coordinate system via $x = r\cos\phi$, $y = r\sin\phi$, and $z = z$, where ϕ is the azimuthal angle. An adaptive rectangular mesh is introduced for the domain of the surface, see figure 2. Each component of the electric field and its normal derivative is assumed to be constant over each rectangle. JDAY numerically computes the surface integrals that are contained in (14) and (22) by setting the observation point at the centroid of each rectangle. After all the surface integrals have been computed, JDAY forms a linear system of algebraic equations that it solves for the boundary unknowns using an iterative method. The iterative method used by JDAY is called the Generalized Minimum Residual method, which was provided by Numerical Algorithms Group Ltd. Once the boundary unknowns are found, JDAY uses them to compute the scattered field in region one by numerically evaluating (11).

[1] JDAY is available from the authors free of charge under the BSD license.

We ran JDAY for six different cases. In each case, the incident wave is given by

$$\mathbf{E}^{(\mathrm{inc})} = (\mathrm{e}^{\mathrm{i}(k_{\mathrm{inc}}z-\omega t)}, 0, 0), \tag{24}$$

where $k_{\mathrm{inc}} = \sqrt{\epsilon_1\mu_1}\omega$, $\omega = 1$, $\epsilon_1 = 1$, and $\mu_1 = 1$. In cases one to four, we plotted the radar cross-section (RCS) in the xz-plane vs. the scattering angle, θ_{scat}, and compared the result to the Mie series solution obtained using MATLAB$^{\mathrm{TM}}$ functions provided by Mätzler [6]. We define the RCS by

$$\mathrm{RCS} = 4\pi R_\infty^2 \frac{E_i^s \overline{E_i^s}}{E_i^{\mathrm{inc}} \overline{E_i^{\mathrm{inc}}}}, \tag{25}$$

where $\bar{}$ denotes the complex conjugate and R_∞ is taken to be 16 incident wavelengths away from the centre of the scatterer, which is a sphere with radius R. Case one, where electric size parameter $k_{\mathrm{inc}}R = 1$, $\epsilon = 4$, $\mu = 1$ is shown in figure 3. Figure 3 illustrates that the solution obtained via JDAY is in good agreement with the Mie series solution, especially for non-forward scattering, i.e. $\theta_{\mathrm{scat}} \neq 0°$. Case two, where electric size parameter $k_{\mathrm{inc}}R = 1$, $\epsilon = 2$, $\mu = 2$ is shown in figure 4. Figure 4 illustrates that the solution obtained via JDAY is in good agreement with the Mie series solution and that both the JDAY and Mie series solutions satisfy

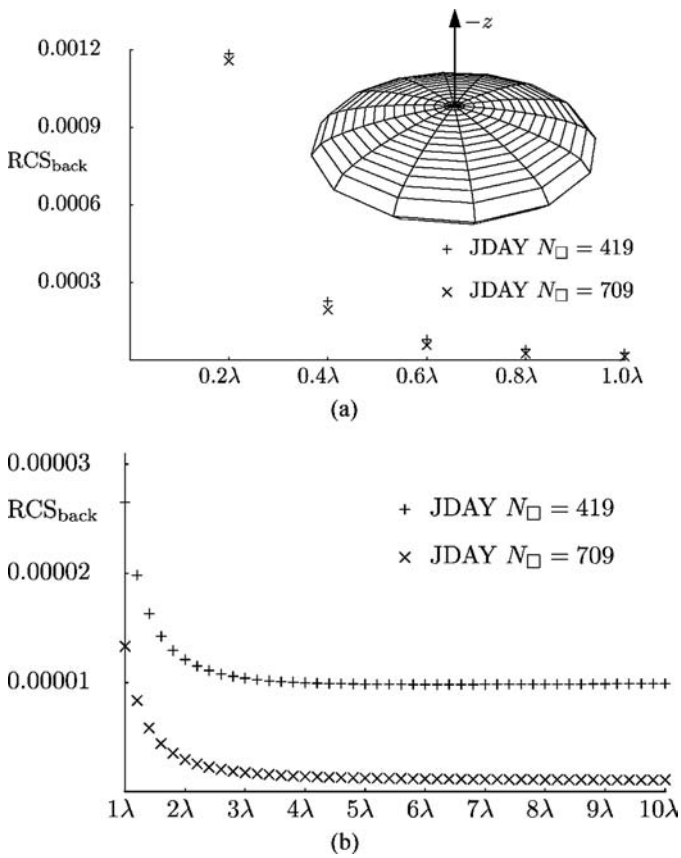

Figure 7. Case 5: $\epsilon = 2$, $\mu = 2$. In figure 7(a) and 7(b), $\mathrm{RCS}_{\mathrm{back}}$ is plotted as a function of distance, measured in numbers of incident wavelengths, from the lowest surface point; i.e. (0, 0, 0.2). $2N_\square$ surface patches were used to obtain the JDAY solution. The scattering surface is shown in figure 7(a).

Weston's theorem [7]. For the reader's convenience, we state Weston's theorem here without proof.

THEOREM 1 *If a plane electromagnetic wave is incident upon a body comprised of material such that $\epsilon = \mu$, then the far zone back-scattered field is zero, provided that the direction of incident propagation is parallel to an axis of the body about which a rotation of* 90° *leaves the shape of the body together with its material medium invariant.*

Case three, where electric size parameter $k_{\rm inc}R = 1$, $\epsilon = 1$, $\mu = 4$ is shown in figure 5. Case four, where electric size parameter $k_{\rm inc}R = 1$, $\epsilon = 1 + 4\mathrm{i}$, $\mu = 1$ is shown in figure 6. In cases three and four, the JDAY solution agrees well with the Mie series solution. In cases five and six, we plotted the back-scattered ($\theta_{\rm scat} = 180^\circ$) radar cross-sections $\mathrm{RCS}_{\rm back}$ vs. distance away from the scatterer, where $\mathrm{RCS}_{\rm back}$ is given by

$$\mathrm{RCS}_{\rm back} = 4\pi z^2 \frac{E_i^s \overline{E_i^s}}{E_i^{\rm inc}\overline{E_i^{\rm inc}}}. \tag{26}$$

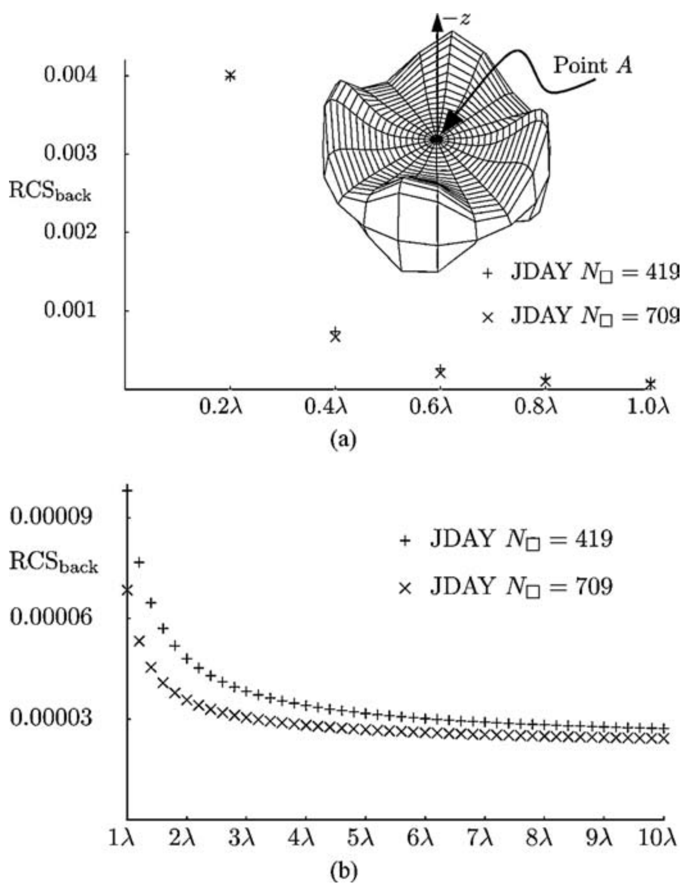

Figure 8. Case 6: $\epsilon = 2$, $\mu = 2$. In figure 8(a) and 8(b), $\mathrm{RCS}_{\rm back}$ is plotted as a function of distance, measured in numbers of incident wavelengths, from point A, see figure 8(a). $2N_\square$ surface patches were used to obtain the JDAY solution. The scattering surface is shown in figure 8(a).

Case five, where $\epsilon = 2$, $\mu = 2$, and the scattering surface is given by

$$S^{\pm} = \pm 0.2\sqrt{1 - r^2}, \tag{27}$$

is shown in figure 7. Figure 7 illustrates the numerical convergence of the solution to the result predicted by Weston's theorem. In case six, $\epsilon = 2$, $\mu = 2$, and the scattering surface is given by

$$S^{\pm} = \pm 2[r^4 \sin^2(2\phi) + 0.1]\sqrt{1 - r^2}. \tag{28}$$

Notice that the surface given by (28) is not a surface of revolution; however, it does satisfy the conditions of Weston's theorem. Figure 8 illustrates the numerical convergence of the solution to the result predicted by Weston's theorem.

4. Conclusions

A new integral equation method for direct electromagnetic scattering in homogeneous media was developed by choosing an electric field and its normal derivative as the boundary unknowns. In the derivation of the new integral equation method, we used a generalized version of the continuity conditions for the electric field and its normal derivative that were originally derived by DeSanto in [1]. The new integral equation method leads to an impedance (collocation) matrix that is 50% sparse. The sparsity of the impedance matrix may offer significant computational advantages over the standard Stratton–Chu integral equation method, which leads to a 100% dense impedance (collocation) matrix.

JDAY, a computer program, was written in order to numerically confirm the validity of the new integral equation method. By running several numerical simulations using JDAY, we were able to confirm the validity of our new integral equation method for a number of different surfaces. We hope that our new integral equation method will be adopted by the electromagnetic community; we therefore offer the JDAY source code free of charge under the BSD license.

Acknowledgments

This paper is based on a thesis submitted by A. Yuffa in partial fulfillment of the requirements for the degree of Masters of Science in Applied Physics at Colorado School of Mines. A. Yuffa is now with Numerica Corporation, Fort Collins, CO 80525, USA.

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Appendix A: Continuity conditions

The usual continuity conditions for the electric field are given by

$$\mathbf{n}^{\pm} \cdot \epsilon_2 \mathbf{E}^{(2)} = \mathbf{n}^{\pm} \cdot \epsilon_1 \mathbf{E}^{(1)} \tag{29}$$

$$\mathbf{n}^{\pm} \times \mathbf{E}^{(2)} = \mathbf{n}^{\pm} \times \mathbf{E}^{(1)}, \tag{30}$$

where the permittivity, ϵ, is a complex constant in the case of an uncharged ohmic conductor. The algebraic linear system of four scalar equations given by (29) and (30) only has three linear independent scalar equations. The system can be solved by using the Moore–Penrose pseudo-inverse to obtain

$$E_i^{(2)} = C_{ij} E_j^{(1)} \tag{31}$$

with

$$C_{ij} = \delta_{ij} + (\epsilon^{-1} - 1)\hat{n}_i^{\pm}\hat{n}_j^{\pm}, \tag{32}$$

where $\epsilon = \epsilon_2/\epsilon_1$, $\hat{\mathbf{n}}^{\pm}$ is a unit normal to $S^{\pm}$, and δ_{ij} is the Kronecker delta function.

To derive the continuity condition for the normal derivative of the electric field, we make use of the following usual boundary condition,

$$\mathbf{n}^{\pm} \times \mathbf{H}^{(2)} = \mathbf{n}^{\pm} \times \mathbf{H}^{(1)}, \tag{33}$$

and two Maxwell's equations given by

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \tag{34}$$

$$\nabla \times \mathbf{E} = \mathrm{i}\omega\mu\mathbf{H}, \tag{35}$$

where the harmonic time factor and region number have been suppressed. Implicitly, (29) and (33) state that there is no net surface charge on the scatterer [8]. Equation (34) even holds on the surface of the scatterer because there is no excess surface charge. We form a vector-cross product of (35) with $\mathbf{n}^{\pm}$ to obtain

$$n_j^{\pm}\partial_j' E_i(\mathbf{x}'_{S^{\pm}}) = n_j^{\pm}\partial_i' E_j(\mathbf{x}'_{S^{\pm}}) + i\omega\mu K_i^{\pm}, \tag{36}$$

where $\mathbf{K}^{\pm} = -\mathbf{n}^{\pm} \times \mathbf{H}^{\pm}$. It is clear that by $\partial_i' E_j(\mathbf{x}'_{S^{\pm}})$, we mean the following order of operations:

(1) Differentiate the electric field as a function of the source variable $\mathbf{x}'$ off the surface.
(2) Evaluate the differentiated electric field on the surface, i.e. $\mathbf{x}' \to \mathbf{x}'_{S^{\pm}}$.

We want to use the opposite order of operations:

(1) Evaluate the electric field on the surface $S^{\pm}(x', y')$.
(2) Differentiate the electric field as a function defined on the surface, i.e. in terms of x' and y'.

It is clear that we cannot simply use the new order of operations. Thus, we must find a relationship between the two different orders of operations. In order to eliminate a possible source of confusion, we define the curly bracket notation

$$\{E_j\} = E_j(x', y', S^{\pm}(x', y')). \tag{37}$$

Thus, the transverse derivative of $\left\{E_j\right\}$ is given by

$$\partial_{it}'\{E_j\} = \{\partial_{it}' E_j\} + \{\partial_3' E_j\}\partial_{it}' S^{\pm}(x', y'). \tag{38}$$

Adding $\delta_{i3}\{\partial_3' E_j\}$ to both sides of (38) yields

$$\{\partial_i' E_j\} = \partial_{it}'\{E_j\} \pm n_i^{\pm}\{\partial_3' E_j\}. \tag{39}$$

Rewriting (36) in the curly bracket notation and substituting (39) into it yields

$$n_j^{\pm}\{\partial_j' E_i\} = n_j^{\pm}\partial_{it}'\{E_j\} \pm n_i^{\pm} n_j^{\pm}\big\{\partial_3' E_j\big\} + \mathrm{i}\omega\mu K_i^{\pm}. \tag{40}$$

The first term on the right-hand side (RHS) of (40) can be integrated by parts because it will appear in the surface integral. However, the second term on the RHS of (40) cannot be integrated by parts, thus, we need to develop it further. To develop it, we substitute the self-evident identity

$$\{\partial_{pt}' E_{pt}\} = \partial_{pt}'\{E_{pt}\} \pm n_{pt}^{\pm}\{\partial_3' E_{pt}\} \tag{41}$$

into (34) to obtain

$$\{\partial_3' E_3\} = -\partial_{pt}'\{E_{pt}\} \mp n_{pt}^{\pm}\{\partial_3' E_{pt}\}. \tag{42}$$

Writing $n_j^{\pm}\{\partial_3' E_j\}$ explicitly and substituting (42) into it yields

$$n_j^{\pm}\{\partial_3' E_j\} = \mp\partial_{pt}'\{E_{pt}\}. \tag{43}$$

Substituting (43) into (40) yields

$$n_j^{\pm}\big\{\partial_j' E_i\big\} = n_j^{\pm}\partial_{it}'\big\{E_j\big\} - n_i^{\pm}\partial_{pt}'\big\{E_{pt}\big\} + \mathrm{i}\omega\mu K_i^{\pm}. \tag{44}$$

Finally, writing (44) for regions one and two, then taking the difference between them and using (33) yields

$$\begin{aligned} n_j^{\pm}\big\{\partial_j' E_i^{(2)}\big\} = \mu\big(& n_j^{\pm}\big\{\partial_j' E_i^{(1)}\big\} - n_j^{\pm}\partial_{it}'\big\{E_j^{(1)}\big\} + n_i^{\pm}\partial_{pt}'\big\{E_{pt}^{(1)}\big\}\big) \\ & + n_j^{\pm}\partial_{it}'\big\{E_j^{(2)}\big\} - n_i^{\pm}\partial_{pt}'\big\{E_{pt}^{(2)}\big\}, \end{aligned} \tag{45}$$

where $\mu = \mu_2/\mu_1$. Observe that the last four terms on the RHS of (45) can be integrated by parts when they appear in the surface integral. Once the terms are integrated by parts, only the electric field and its normal derivative will appear in the integrand.

The derivation of (31) and (45) has been adapted from [1]. Please note that the first term on the left-hand side of equation (61) on page 1299 in [1] should read $n_j\{\partial_j E_i^{(2)}\}$ and not $n_j\partial_j\{E_i^{(2)}\}$.