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PAPER

Vectorizing Green's identities

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Abstract

Green's theorem and Green's identities are well-known and their uses span almost every branch of science and mathematics. In this paper, we derive a vector analogue of Green's three scalar identities and consider some of their uses. We also offer a number of historical tidbits in connection to the work of George Green.

1. Introduction

In 2028, it will be exactly two centuries since George Green, a self-taught miller of Nottingham, wrote his revolutionary essay titled *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* [1]. In the first part of the essay, Green writes:

> Let U and V be two continuous functions of the rectangular co-ordinates x, y, z, whose differential co-efficients do not become infinite at any point within a solid body of any form whatever; then will

$$\int dx\ dy\ dz\ U\delta V + \int d\sigma\ U\left(\frac{dV}{dw}\right) = \int dx\ dy\ dz\ V\delta U + \int d\sigma V\left(\frac{dU}{dw}\right);$$

the triple integrals extending over the whole interior of the body, and those relative to $d\sigma$, over its surface, of which $d\sigma$ represents and element: dw being an infinitely small line perpendicular to the surface, and measured from this surface towards the interior of the body [δ denotes the Laplacian operator $\triangle$]. [1], p. 23The above equation, which Green proves via integration by parts, is what we now call *Green's second identity* or *Green's theorem*. The importance of Green's second identity in the exact sciences cannot be overstated. Without a doubt, the reader of this journal has seen Green's second identity in at least one scientific subfields such as electromagnetism, fluid mechanics, acoustics, quantum field theory or even the theory of functions of a complex variable. However, the reader may be unaware of the vector analogue of Green's second identity even though it has been available since at least 1913 [2], p. 182. Recently, Fernández-Guasti pointed out that the vector version of Green's second identity is not readily available even in specialized literature [3]. To remedy this situation, Fernández-Guasti performs a rather cumbersome derivation of Green's second vector identity by explicitly restricting the derivation to a Cartesian coordinate system [3].

In this paper, we derive Green's second vector identity as a natural consequence of his first vector identity. Green's third vector identity can be derived from his second identity but the derivation is somewhat sophisticated because it requires the covariant derivatives to commute. Therefore, to make this paper appealing to a larger audience, we present a simple alternative derivation in the body of the paper and defer the sophisticated derivation of Green's third vector identity to the Appendix. In order not to restrict Green's three vector identities to a Cartesian coordinate system, we use some elementary elements of tensor calculus that should be familiar to the reader from a typical math-methods course or electrodynamics course, e.g., see [4], Ch. 2 and [5], Ch. 11, respectively. Having established a vector analogue of Green's three identities, we consider their application to the vector Laplace and Helmholtz equations [6]. Furthermore, we point out some erroneous

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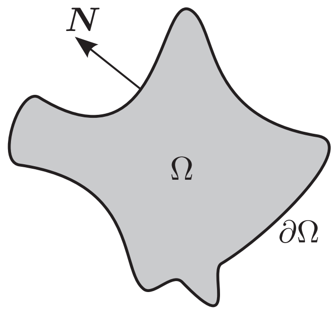

Figure 1. Volume Ω with boundary $\partial\Omega$ and an outward unit-normal **N** are shown.

and/or misleading statements in the literature in connection with Green's second vector identity and vector diffraction theory.

2. Background review and notation

We only consider a simply connected finite volume Ω with a smooth boundary $\partial\Omega$ and denote the differential volume and differential surface element by $\mathrm{d}^3\Omega$ and $\mathrm{d}^2\Omega$, receptively. The outward unit normal to $\partial\Omega$ is denoted by $\mathbf{N}$, see figure 1.

The coordinates of a source point in the three-dimensional Euclidean space are denoted by Z^1, Z^2, Z^3 or simply by Z, and the position vector is denoted by $\mathbf{r} = \mathbf{r}(Z)$. Thus, the covariant ambient basis vectors are given by

$$\mathbf{Z}_i = \frac{\partial}{\partial Z^i}\mathbf{r}(Z) \qquad \text{for } i = 1, 2, 3.$$

For example, in the usual Cartesian coordinate system (x, y, z) we simply have $\mathbf{Z}_1 = \hat{\mathbf{x}}$, $\mathbf{Z}_2 = \hat{\mathbf{y}}$, $\mathbf{Z}_3 = \hat{\mathbf{z}}$ and in the spherical coordinate system (r, θ, ϕ) we have

$$\begin{aligned}
\mathbf{Z}_1 &= \sin\theta\cos\phi\ \hat{\mathbf{x}} + \sin\theta\sin\phi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}} \equiv \hat{\mathbf{r}},\\
\mathbf{Z}_2 &= r\cos\theta\cos\phi\ \hat{\mathbf{x}} + r\cos\theta\sin\phi\ \hat{\mathbf{y}} - r\sin\theta\ \hat{\mathbf{z}} \equiv r\hat{\boldsymbol{\theta}},\\
\mathbf{Z}_3 &= -r\sin\theta\sin\phi\ \hat{\mathbf{x}} + r\sin\theta\cos\phi\ \hat{\mathbf{y}} \equiv r\sin\theta\hat{\boldsymbol{\phi}}.
\end{aligned}$$

Throughout the paper, we use tensor notation with the Einstein summation convention, where the Latin alphabet indices range from 1 to 3. In this notation, an arbitrary vector $\mathbf{A}$ may be written as $\mathbf{A} = A^i\mathbf{Z}_i = A_i\mathbf{Z}^i$, where A^i and A_i are the contravariant and the covariant components of the vector $\mathbf{A}$. The covariant basis $\mathbf{Z}_i$ are related to the contravariant basis $\mathbf{Z}^i$ via $\mathbf{Z}_i = Z_{ij}\mathbf{Z}^j$, where $Z_{ij} = \mathbf{Z}_i \cdot \mathbf{Z}_j$ is the fundamental tensor. For example, in a Cartesian coordinate system, the fundamental tensor is simply the identity matrix and in the spherical coordinate system it is given by

$$Z_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}.$$

The covariant derivative is denoted by ∇_i and the contravariant derivative by ∇^i. Of course, the two are related via $\nabla_i = Z_{ij}\nabla^j$ and in the Cartesian coordinate system, we have $\nabla_1 = \nabla^1 = \partial/\partial x$, $\nabla_2 = \nabla^2 = \partial/\partial y$, $\nabla_3 = \nabla^3 = \partial/\partial z$. The divergence of the vector field $\mathbf{A}$ is given by $\nabla \cdot \mathbf{A} = \nabla_i A^i = \nabla^i A_i$ and the Laplacian by $\triangle\mathbf{A} = \nabla_i\nabla^i\mathbf{A}$. An important property of the covariant and contravariant derivatives is that they are metrinilic with respect to the basis, i.e., $\nabla_i\mathbf{Z}_j = \mathbf{0}$ and $\nabla^i\mathbf{Z}_j = \mathbf{0}$. The divergence theorem is given by

$$\int_\Omega \nabla_i A^i\ \mathrm{d}^3\Omega = \int_{\partial\Omega} N_i A^i\ \mathrm{d}^2\Omega, \tag{1}$$

where N_i are the covariant components of the outward unit vector $\mathbf{N}$, see figure 1. We will make extensive use of the divergence theorem in the derivations and applications of Green's vector identities. Thus, it is of benefit to consider a simple intuitive example. Intuitively it is clear that $\int_{\partial\Omega}\mathbf{N}\ \mathrm{d}^2\Omega = \mathbf{0}$ for any closed surface $\partial\Omega$ because a body placed in a uniform pressure field remains at rest. To see this mathematically, we apply (1) and use the metrinilic property to obtain

$$\int_{\partial\Omega}\mathbf{N}\ \mathrm{d}^2\Omega = \int_{\partial\Omega} N^i\mathbf{Z}_i\ \mathrm{d}^2\Omega = \int_\Omega \nabla^i\mathbf{Z}_i\ \mathrm{d}^3\Omega = \mathbf{0}.$$

3. Green's first vector identity

To derive Green's first vector identity, we look at the $U \triangle V$ term on the left-hand side (LHS) of Green's second identity quoted in section 1 and consider a relatively obvious ansatz $\mathbf{P} \cdot \triangle \mathbf{Q}$. Substituting the well-known vector identity $\triangle \mathbf{Q} = \nabla(\nabla \cdot \mathbf{Q}) - \nabla \times \nabla \times \mathbf{Q}$ into the ansatz and using the vector identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$ with $\mathbf{A} = \mathbf{P}$ and $\mathbf{B} = \nabla \times \mathbf{Q}$ yields

$$\mathbf{P} \cdot \triangle \mathbf{Q} = \nabla \cdot (\mathbf{P} \times \nabla \times \mathbf{Q}) - (\nabla \times \mathbf{P}) \cdot (\nabla \times \mathbf{Q}) + \mathbf{P} \cdot [\nabla(\nabla \cdot \mathbf{Q})]. \tag{2}$$

The last term on the right-hand side (RHS) of (2) is the root cause of the agony in the derivation by Fernández-Guasti [3]. To avoid the agony, we re-write it in tensor notation and use the product rule to obtain

$$\begin{aligned} \mathbf{P} \cdot [\nabla(\nabla \cdot \mathbf{Q})] &= P^i[\nabla_i(\nabla_j Q^j)] \\ &= \nabla_i[P^i(\nabla_j Q^j)] - (\nabla_i P^i)(\nabla_j Q^j) \\ &= \nabla \cdot [\mathbf{P}(\nabla \cdot \mathbf{Q})] - (\nabla \cdot \mathbf{P})(\nabla \cdot \mathbf{Q}). \end{aligned} \tag{3}$$

Finally, substituting (3) into (2) and using the divergence theorem we obtain Green's first vector identity; namely,

$$\begin{aligned} &\int_{\partial\Omega} \mathbf{N} \cdot [\mathbf{P} \times \nabla \times \mathbf{Q} + \mathbf{P}(\nabla \cdot \mathbf{Q})] \, d^2\Omega \\ &= \int_{\Omega} \mathbf{P} \cdot \triangle \mathbf{Q} \, d^3\Omega + \int_{\Omega} [(\nabla \times \mathbf{P}) \cdot (\nabla \times \mathbf{Q}) + (\nabla \cdot \mathbf{P})(\nabla \cdot \mathbf{Q})] \, d^3\Omega. \end{aligned} \tag{4}$$

Two useful alternative forms of the LHS of (4) can be obtained via the vector identities $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$, i.e., $\mathbf{N} \cdot [\mathbf{P} \times \nabla \times \mathbf{Q}] = -\mathbf{P} \cdot [\mathbf{N} \times \nabla \times \mathbf{Q}] = (\mathbf{N} \times \mathbf{P}) \cdot (\nabla \times \mathbf{Q})$.

If we set $\mathbf{Q} = \mathbf{P}$ in (4), then the last integrand on the RHS of (4) is nonnegative. Thus, we can refer to the last integral on the RHS of (4) as the Dirichlet energy and expect it to play an essential role in the uniqueness proofs of (vector) Poisson's equation.

4. Green's second and third vector identities

Green's second (scalar) identity is traditionally derived from his first identity by subtracting from the first identity another first identity with the interchanged roles of U and V. The same procedure can be used to derive Green's second vector identity; interchanging the roles of $\mathbf{P}$ and $\mathbf{Q}$ in (4) and subtracting it from (4) immediately yields

$$\begin{aligned} \int_{\Omega} [\mathbf{Q} \cdot \triangle \mathbf{P} - \mathbf{P} \cdot \triangle \mathbf{Q}] d^3\Omega = &\int_{\partial\Omega} \mathbf{N} \cdot [\mathbf{Q} \times \nabla \times \mathbf{P} - \mathbf{P} \times \nabla \times \mathbf{Q}] d^2\Omega \\ &+ \int_{\partial\Omega} [(\mathbf{N} \cdot \mathbf{Q})(\nabla \cdot \mathbf{P}) - (\mathbf{N} \cdot \mathbf{P})(\nabla \cdot \mathbf{Q})] d^2\Omega. \end{aligned} \tag{5}$$

Green's third (scalar) identity is ordinarily derived by letting U or V in the second identity be the free-space Green's function satisfying

$$\triangle g(\mathbf{r}', \mathbf{r}) = -\delta(\mathbf{r}' - \mathbf{r}), \tag{6}$$

where $\delta(\mathbf{r}' - \mathbf{r})$ is the Dirac delta function, $\mathbf{r}' = \mathbf{r}(Z')$ is the position vector of an observation (field) point, and $\triangle$ operates on the source coordinates Z (not Z'). We present this approach in the appendix because we prefer to show a much simpler approach here based directly on (6). To derive Green's third vector identity from (6), we multiply (6) by $\mathbf{P}$ and subtract $g\nabla_i\nabla^i\mathbf{P}$ from both sides of the resultant equation to obtain

$$\nabla_i(\mathbf{P}\nabla^i g - g\nabla^i\mathbf{P}) = -\mathbf{P}\delta(\mathbf{r}' - \mathbf{r}) - g \triangle \mathbf{P}. \tag{7}$$

Integrating (7) over the volume Ω and using the divergence theorem on the LHS of (7) yields Green's third vector identity

$$\int_{\partial\Omega} \left(g\frac{\partial \mathbf{P}}{\partial N} - \mathbf{P}\frac{\partial g}{\partial N} \right) d^2\Omega - \int_{\Omega} g \triangle \mathbf{P} \, d^3\Omega = \begin{cases} \mathbf{P}(\mathbf{r}') & for \, \mathbf{r}' \in \Omega \\ \mathbf{0} & for \, \mathbf{r}' \notin \Omega \end{cases}, \tag{8}$$

where the normal derivative is denoted by $\partial/\partial N = \mathbf{N} \cdot \nabla = N^i\nabla_i$. From (8), we see that the knowledge of $\mathbf{P}$ and $\partial\mathbf{P}/\partial N$ on the boundary $\partial\Omega$ is sufficient to reconstruct a vector harmonic function (a function that satisfies $\triangle \mathbf{P} = \mathbf{0}$) inside the volume Ω.

5. Application of Green's first identity

It is common to use Green's first (scalar) identity in the uniqueness proofs of a solution to (scalar) Poisson's equation with the Dirichlet and the Neumann boundary conditions. To keep with our parallel development of the vector Green's identities, we present a similar analysis of the vector Poisson's equation, where the proof is a bit more intricate and the space of the boundary conditions is a bit more vast.

Let $\mathbf{U}$ be a solution to the vector Poisson's equation

$$\triangle \mathbf{U} = \mathbf{F} \quad \text{in}\Omega \tag{9}$$

with the Dirichlet boundary conditions

$$\mathbf{U} = \mathbf{L} \quad \text{on}\partial\Omega, \tag{10a}$$

the magnetic boundary conditions

$$\mathbf{N} \times \nabla \times \mathbf{U} = \mathbf{J} \qquad \text{and} \qquad \mathbf{N} \cdot \mathbf{U} = \sigma \quad \text{on}\partial\Omega, \tag{10b}$$

or the electric boundary conditions

$$\mathbf{N} \times \mathbf{U} = \mathbf{K} \qquad \text{and} \qquad \nabla \cdot \mathbf{U} = \rho \quad \text{on}\partial\Omega. \tag{10c}$$

If we assume there are two solutions $\overset{1}{\mathbf{U}}$ and $\overset{2}{\mathbf{U}}$ that satisfy (9) with one of the boundary conditions in (10), then $\mathbf{V} = \overset{1}{\mathbf{U}} - \overset{2}{\mathbf{U}}$ satisfies the vector Laplace equation

$$\triangle \mathbf{V} = \mathbf{0} \quad \text{in}\Omega \tag{11}$$

with the homogeneous boundary conditions:

$$\mathbf{V} = \mathbf{0} \quad \text{on}\partial\Omega, \tag{12a}$$

$$\mathbf{N} \times \nabla \times \mathbf{V} = \mathbf{0} \qquad \text{and} \qquad \mathbf{N} \cdot \mathbf{V} = 0 \quad \text{on}\partial\Omega, \tag{12b}$$

or

$$\mathbf{N} \times \mathbf{V} = \mathbf{0} \qquad \text{and} \qquad \nabla \cdot \mathbf{V} = 0 \quad \text{on}\partial\Omega. \tag{12c}$$

From (*12b*) and (*12c*), we see that the terms magnetic and electric boundary conditions come from the boundary conditions that the electric field satisfies on the surface of a perfect magnetic and electric conductor, respectively [7]. If we set $\mathbf{P} = \mathbf{Q} = \mathbf{V}$ in the Green's first vector identity (4) and use (11) with (*12a*), (*12b*) or (*12c*), then we see that the Dirichlet energy vanishes in all three cases, i.e.,

$$\int_\Omega (|\nabla \times \mathbf{V}|^2 + |\nabla \cdot \mathbf{V}|^2) \mathrm{d}^3\Omega = 0. \tag{13}$$

The functions $|\nabla \times \mathbf{V}|$ and $|\nabla \cdot \mathbf{V}|$ are nonnegative. Therefore, the only way the integral in (13) can vanish is if we have

$$\nabla \times \mathbf{V} = \mathbf{0} \qquad \text{and} \qquad \nabla \cdot \mathbf{V} = 0 \quad \text{in}\Omega. \tag{14}$$

From the first equation in (14), we see that $\mathbf{V}$ can be written as a gradient of the scalar potential ψ and, from the second equation in (14), we see that ψ satisfies Laplace's equation, i.e., $\triangle\psi = 0$ in Ω. From (*12a*) and (*12b*) it follows that in the case of the homogeneous Dirichlet and magnetic boundary conditions we have the homogeneous Neumann boundary condition on ψ, i.e., $\partial\psi/\partial N = 0$ on $\partial\Omega$. It is well-known from Green's first (scalar) identity that the Neumann problem is unique up to a constant. Thus, $\mathbf{V} = \mathbf{0}$ and the solution (if it exists) to the vector Poisson's equation is unique provided we have the Dirichlet or magnetic boundary conditions.

In the case of the homogeneous electric boundary conditions, the situation is more difficult because after applying Green's first (scalar) identity we obtain

$$\int_\Omega |\nabla\psi|^2 \mathrm{d}^3\Omega = \int_{\partial\Omega} \psi \frac{\partial\psi}{\partial N} \mathrm{d}^2\Omega \tag{15}$$

instead of $\int_\Omega |\nabla\psi|^2 \mathrm{d}^3\Omega = 0$. To proceed further, we decompose the gradient operator into the tangential and normal parts to obtain

$$\nabla = \sum_{\alpha=1}^{2} \mathbf{S}_\alpha \nabla^\alpha + \mathbf{N}\frac{\partial}{\partial N}, \tag{16}$$

where $\mathbf{S}_\alpha$ are the covariant basis tangential to the surface $\partial\Omega$ and ∇^α is the surface contravariant derivative. Substituting (16) into the first equation in (*12c*) and noting that $\mathbf{S}_\alpha$ and $\mathbf{N}$ are orthogonal yields

$$\sum_{\alpha=1}^{2} \mathbf{S}_\alpha \nabla^\alpha \psi = \mathbf{0}. \tag{17}$$

From (17) we see that ψ is constant on the surface $\partial\Omega$, and therefore, ψ may be taken out of the surface integral in (15) to obtain

$$\int_\Omega |\nabla\psi|^2 d^3\Omega = \psi \int_{\partial\Omega} \frac{\partial\psi}{\partial N} d^2\Omega. \tag{18}$$

After applying the divergence theorem to the second equation in (14) we see that the surface integral of the normal derivative of ψ vanishes. Therefore, the RHS of (18) also vanishes and we conclude that ψ must be constant in Ω. Thus, $\mathbf{V} = \mathbf{0}$ in Ω and we see that the solution (if it exists) to the vector Poisson's equation with the electric boundary conditions is unique.

6. Application of Green's second and third vector identities

Perhaps one of the most important theorems in electromagnetic wave diffraction or in antenna theory is the Stratton–Chu formula [8]. This formula is sometimes also called the vector Kirchhoff formula, presumably because it can be directly derived from the scalar Kirchhoff formula. However, Baker and Copson object to such a derivation;

> To deal with radiation, it is in fact necessary to have recourse to the electromagnetic theory of light. Whilst it is true that Kirchhoff's formula can be applied to each of the components of the electric and magnetic vectors, this does not constitute a valid formulation of Huygens' principle as it possesses no physical interpretation. [9], p. 102

Baker and Copson derive what they call the Larmor–Tedone formula (Stratton–Chu formula) by following E. T. Whittaker's notes, which were presumably based on the work of Larmor and Tedone. Judging by the citation on page 106 in [9] it does indeed seem that Larmor and Tedone derived the formula circa 1917, i.e., over two decades before Stratton and Chu. Tai points this out and suggests that:

> [...] presumably because of the simplicity with which Stratton and Chu derived their formula, the formula bears the names of the two authors. [10], p. 104

However, this is doubtful because eight years earlier Murray [11] independently derived the Stratton–Chu formula via a simple application of Green's second vector identity. Furthermore, judging from the work of Hasenöhrl [12], it is reasonable to assume that at least some form of the Stratton–Chu formula was known since 1906.

In modern literature, there is still a disconnect between the Stratton–Chu formula derived via Green's second vector identity and via the Kirchhoff formula. The derivations are unnecessarily restricted to a Cartesian coordinate system and thereby implicitly or explicitly convey to the reader that the two are only equivalent in a Cartesian coordinate system; e.g., see [5], §10.6 for the Cartesian restricted derivation of the Stratton–Chu formula from the scalar Kirchhoff formula and [13] for the Cartesian restricted reduction of the Stratton–Chu formula to the Kirchhoff formula. We address this disconnect below.

Assume that all fields are harmonic in time with the suppressed time factor $\exp(-i\omega t)$, where ω is the angular frequency, t denotes time, and $i \equiv \sqrt{-1}$. Furthermore, if we assume the source-free volume Ω can be characterized by the complex constants ϵ (permittivity) and μ (permeability), then from Maxwell's equations we have

$$\nabla \times \mathbf{E} = i\mu\omega\mathbf{H} \quad \text{in } \Omega \text{ and on } \partial\Omega, \tag{19a}$$

$$\nabla \cdot \mathbf{E} = 0 \quad \text{in } \Omega \text{ and on } \partial\Omega, \tag{19b}$$

and

$$\triangle\mathbf{E} + k^2\mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \tag{19c}$$

where the wavenumber $k = \sqrt{\epsilon\mu}\,\omega$, $\mathbf{E}$ denotes the electric field, and $\mathbf{H}$ denotes the magnetic field. The free-space Green's function, $G(\mathbf{r}', \mathbf{r})$, to the vector Helmholtz equation (*19c*) is defined by

$$\triangle G(\mathbf{r}', \mathbf{r}) + k^2 G(\mathbf{r}', \mathbf{r}) = -\delta(\mathbf{r}' - \mathbf{r}). \tag{20}$$

Similarly to our derivation of Green's third vector identity in section 4, we multiply (*19c*) by G and (20) by $\mathbf{E}$, then apply the divergence theorem to the difference to obtain

$$\int_{\partial\Omega}\left(G\frac{\partial\mathbf{E}}{\partial N}-\mathbf{E}\frac{\partial G}{\partial N}\right)\mathrm{d}^2\Omega=\begin{cases}\mathbf{E}(\mathbf{r}') & for\,\mathbf{r}'\in\Omega\\ \mathbf{0} & for\,\mathbf{r}'\notin\Omega\end{cases}. \tag{21}$$

Notice that (21) is essentially Green's third vector identity with the free-space Green's function for the Helmholtz equation instead of for the Laplace equation, i.e., $G(\mathbf{r}', \mathbf{r})$ instead of $g(\mathbf{r}', \mathbf{r})$. Furthermore, notice that in a Cartesian coordinate system, (21) yields the scalar Kirchhoff formula for each rectangular component; however, (21) is *not* restricted to the Cartesian coordinate system.

It is difficult to see directly from (21) that it is equivalent to the Stratton–Chu formula in Euclidean space. Thus, we will show this equivalency in an indirect manner. Substituting $\mathbf{P}=\mathbf{E}$ and $\mathbf{Q}=G\mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector, into Green's second vector identity (5), then using 19, (20), and the vector identity

$$\mathbf{N}\cdot\{\mathbf{E}\times[\nabla\times(G\mathbf{C})]\}=\mathbf{C}\cdot[(\mathbf{N}\times\mathbf{E})\times\nabla G]$$

yields the Stratton–Chu formula, namely,

$$-\int_{\partial\Omega}[\mathrm{i}\mu\omega(\mathbf{N}\times\mathbf{H})G+(\mathbf{N}\cdot\mathbf{E})\nabla G+(\mathbf{N}\times\mathbf{E})\times\nabla G]\mathrm{d}^2\Omega=\begin{cases}\mathbf{E}(\mathbf{r}'), & \mathbf{r}'\in\Omega\\ \mathbf{0}, & \mathbf{r}'\notin\Omega\end{cases}. \tag{22}$$

However, substituting $\mathbf{P}=\mathbf{E}$ and $\mathbf{Q}=G\mathbf{C}$ into Green's second vector identity (5), then proceeding as in the appendix yields (21). Therefore, we see that the Stratton–Chu formula is indeed equivalent to (21). Of course, it is possible to directly reduce the Stratton–Chu formula (22) to (21). This is done explicitly in section V of [14]. Finally, we remark that the fact that the covariant derivatives commute (the Riemann–Christoffel tensor vanishes) in Euclidean space is essential to establishing the equivalency between (22) and (21).

7. Summary

We derived three Green's vector identities given by (4), (5), and (8). The derivations were performed in a natural way without restricting ourselves to a Cartesian coordinate system. We used Green's first vector identity to show the uniqueness of a solution to the vector Poisson equation under Dirichlet, magnetic, and electric boundary conditions, see section 5. Furthermore, we used Green's second and third vector identities to address the equivalency of the Stratton–Chu formula and the Kirchoff formula, see section 6.

We would like to end this paper with a quote from Julian Schwinger which best describes George Green and his work:

> What, finally, shall we say about George Green? Why, that he is, in a manner of speaking, alive, well, and living among us. [15], p. 11

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

To derive Green's third vector identity from his second vector identity, we substitute $\mathbf{Q}=g\mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector and g is defined by (6), into (5) to obtain

$$\begin{aligned}\mathbf{C}\cdot\int_{\Omega}(g\bigtriangleup\mathbf{P}-\mathbf{P}\bigtriangleup g)\mathrm{d}^3\Omega&=\int_{\partial\Omega}\mathbf{N}\cdot[g\mathbf{C}\times\nabla\times\mathbf{P}-\mathbf{P}\times\nabla\times(g\mathbf{C})]\mathrm{d}^2\Omega\\&\quad+\mathbf{C}\cdot\int_{\partial\Omega}[(\nabla\cdot\mathbf{P})g\mathbf{N}-(\mathbf{N}\cdot\mathbf{P})\nabla g]\mathrm{d}^2\Omega.\end{aligned} \tag{A.1}$$

To develop the first term on the RHS of (A.1) further, we make use of two vector identities

$$\mathbf{N}\cdot[g\mathbf{C}\times\nabla\times\mathbf{P}]=g\mathbf{C}\cdot\frac{\partial\mathbf{P}}{\partial N}+(\mathbf{N}\cdot\mathbf{P})(\mathbf{C}\cdot\nabla g)-\mathbf{N}\cdot[(\mathbf{C}\cdot\nabla)(g\mathbf{P})] \tag{A.2}$$

and

$$\mathbf{N}\cdot[\mathbf{P}\times\nabla\times(g\mathbf{C})]=(\mathbf{C}\cdot\mathbf{P})\frac{\partial g}{\partial N}+(\mathbf{C}\cdot\mathbf{N})g\nabla\cdot\mathbf{P}-(\mathbf{C}\cdot\mathbf{N})\nabla\cdot(g\mathbf{P}). \tag{A.3}$$

These vector identities are esoteric when written in the Gibbs notation. However, they follow naturally in the tensor notation by writing the cross-products on the LHS in terms of the Levi-Civita symbols and using the well-known identity $\varepsilon_{ijk}\varepsilon^{k\ell m}=\delta^{\ell}{}_{i}\delta^{m}{}_{j}-\delta^{m}{}_{i}\delta^{\ell}{}_{j}$, where ε_{ijk} is the Levi-Civita symbol and $\delta^{i}{}_{j}$ is the Kronecker symbol.

For example, from the LHS of (A.2) we have

$$\mathbf{N}\cdot[g\mathbf{C}\times\nabla\times\mathbf{P}] = gN^i\varepsilon_{ijk}C^j\varepsilon^{k\ell m}\nabla_\ell P_m = g\mathbf{C}\cdot\frac{\partial\mathbf{P}}{\partial N} - gN^iC^j\nabla_j P_i$$

and after using the product rule $g\nabla_j P_i = \nabla_j(gP_i) - P_i\nabla_j g$ we obtain the RHS of (A.2). Substituting (6), (A.2) and (A.3) into (A.1) and noting that the resultant equation must hold for an arbitrary constant vector $\mathbf{C}$ we obtain

$$\int_{\partial\Omega}\left(g\frac{\partial\mathbf{P}}{\partial N} - \mathbf{P}\frac{\partial g}{\partial N}\right)\mathrm{d}^2\Omega - \int_{\Omega} g\,\triangle\,\mathbf{P}\,\mathrm{d}^3\Omega + \mathcal{O} = \begin{cases}\mathbf{P}(\mathbf{r}') & for\,\mathbf{r}'\in\Omega\\ \mathbf{0} & for\,\mathbf{r}'\notin\Omega\end{cases}, \tag{A.4}$$

where

$$\mathcal{O} = \int_{\partial\Omega}(N_i\mathbf{Z}^i\nabla^j - N^j\mathbf{Z}^i\nabla_i)(gP_j)\,\mathrm{d}^2\Omega. \tag{A.5}$$

Finally, using the divergence theorem on the RHS of (A.5) yields

$$\mathcal{O} = \int_{\partial\Omega}\mathbf{Z}^i(\nabla_i\nabla_j - \nabla_j\nabla_i)(gP^j)\,\mathrm{d}^2\Omega = \mathbf{0} \tag{A.6}$$

because the covariant derivatives commute (the Riemann–Christoffel tensor vanishes) in Euclidean space [16], Section 12.6.

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