On Wacker's Essential Equation in the Extrapolation Measurement Technique

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***Abstract*—The generalized three-antenna method is a standard method for measuring on-axis gain and polarization of an antenna without a priori knowledge. The cornerstone of the method is the use of the extrapolation technique and the key relationship in the extrapolation technique is Wacker's equation. This equation expresses the received signal as a function of the separation distance between any two antennas. The derivation of Wacker's equation is not readily available in the literature. In this paper, we provide a streamlined derivation of Wacker's equation and address some of the common misconceptions associated with it.**

I. Introduction

The extrapolation technique is commonly used to measure on-axis gain and polarization of an antenna. This technique was introduced almost five decades ago by Newell, Baird, and Wacker in their seminal paper [1]. The technique is based on an equation that represents the received signal as a function of the separation distance between any two antennas. This equation, see equation (26) in [1], is equally accurate when the antennas are close together or far apart because it accounts for the multiple scattering effects between the two antennas. In the seminal paper, the derivation of the equation is attributed to an unpublished technical report by Wacker. Two years before the seminal paper was published Newell and Kerns [2] attributed the derivation of the equation to another of Wacker's unpublished works with an almost identical title. A decade later Kerns, in his classic monograph [3, p. 148], attributes the derivation of the equation to the unpublished report by Wacker. Furthermore, Kerns cites private communication with Yaghjian (presumably Arthur D. Yaghjian) and states that Yaghjian derived an almost identical equation to Wacker's (26) in [1] via a different method [3, p.148]. At this point the referencing of unpublished literature ends and Kerns gives a brute-force derivation of the equation under a greatly simplifying assumption of no multiple scattering [3, pp. 147–159]. It is not clear why the authors cited the report as unpublished as it was *published* almost a year before the seminal paper was submitted for publication [4]. The front page of the report is shown in Fig. 1 and may be obtain from the National Oceanic and Atmospheric Administration (NOAA) library in Boulder, Colorado.

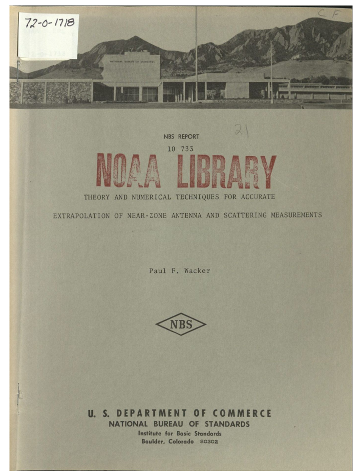

Fig. 1. (Color online) The cover of the allusive Wacker's report [4] is shown.

Perhaps because of the perceived lack of availability of Wacker's report, we have encountered a number of practitioners with a misleading interpretation of Wacker's equation. One misconception is that the distance between the two antennas should be measured from the phase centers of the antennas. Another misconception is that Wacker's equation only *approximately* accounts for the multiple scattering effects. This misconception seems to be caused by the input reflection coefficient of the probe antenna in *free space*. To eliminate some of these misconceptions, we present an abridged derivation of Wacker's equation with the aim of pedagogical clarity.

II. Orders-of-Scattering Interpretation

Consider two on-axis antennas separated by free space as shown in Fig. 2. Without lost of generality, we let the antenna under test (AUT) be the antenna on the left-hand side (LHS) in Fig. 2 and the probe antenna to be on the right-hand side (RHS). The scattering-matrix for each antenna is defined by

$$\begin{bmatrix} b_1^\alpha \\ b_2^\alpha \end{bmatrix} = \begin{bmatrix} S_{11}^\alpha & S_{12}^\alpha \\ S_{21}^\alpha & S_{22}^\alpha \end{bmatrix} \begin{bmatrix} a_1^\alpha \\ a_2^\alpha \end{bmatrix}, \tag{1a}$$

where $\alpha = \ell$ for the LHS antenna and $\alpha = \mathrm{r}$ for the RHS antenna. We can also define a scattering-matrix for the space

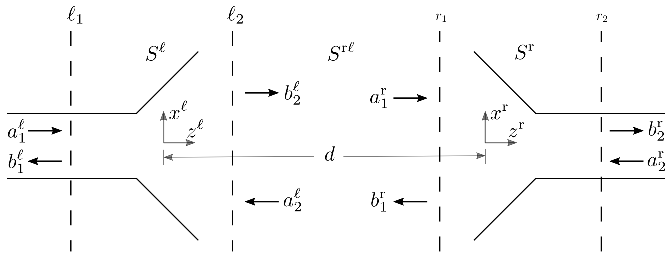

Fig. 2. Two on-axis antennas separated by free space are shown. The terminal surfaces are represented by the labeled dashed lines. The complex amplitudes of the incident and emergent modes are denoted by $a_1^\ell, a_2^{\mathrm{r}}$ and $b_1^\ell, b_2^{\mathrm{r}}$, respectively.

between the antennas via

$$\begin{bmatrix} a_1^{\mathrm{r}} \\ a_2^\ell \end{bmatrix} = \begin{bmatrix} 0 & S_{12}^{\mathrm{r}\ell} \\ S_{21}^{\mathrm{r}\ell} & 0 \end{bmatrix} \begin{bmatrix} b_1^{\mathrm{r}} \\ b_2^\ell \end{bmatrix}, \tag{1b}$$

where the diagonal elements vanish because there is no coupling between the incident and emergent waves at the r_1 terminal surface and the ℓ_2 terminal surface. The scattering-matrix for the whole system is defined by

$$\begin{bmatrix} b_1^\ell \\ b_2^{\mathrm{r}} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1^\ell \\ a_2^{\mathrm{r}} \end{bmatrix} \tag{2}$$

and can be expressed in terms of $S^\ell, S^{\mathrm{r}\ell}, S^{\mathrm{r}}$ by eliminating a_2^ℓ, b_2^ℓ and $a_1^{\mathrm{r}}, b_1^{\mathrm{r}}$ from (1) and comparing the result to (2) to obtain

$$S_{11} = S_{11}^\ell + S_{12}^\ell R^{\mathrm{r}} \left[1 - S_{22}^\ell R^{\mathrm{r}}\right]^{-1} S_{21}^\ell, \tag{3a}$$

$$S_{12} = S_{12}^\ell S_{21}^{\mathrm{r}\ell} \left[1 - S_{11}^{\mathrm{r}} R^\ell\right]^{-1} S_{12}^{\mathrm{r}}, \tag{3b}$$

$$S_{21} = S_{21}^{\mathrm{r}} S_{12}^{\mathrm{r}\ell} \left[1 - S_{22}^\ell R^{\mathrm{r}}\right]^{-1} S_{21}^\ell, \tag{3c}$$

$$S_{22} = S_{22}^{\mathrm{r}} + S_{21}^{\mathrm{r}} R^\ell \left[1 - S_{11}^{\mathrm{r}} R^\ell\right]^{-1} S_{12}^{\mathrm{r}}, \tag{3d}$$

where

$$R^{\mathrm{r}} = S_{21}^{\mathrm{r}\ell} S_{11}^{\mathrm{r}} S_{12}^{\mathrm{r}\ell} \quad \text{and} \quad R^\ell = S_{12}^{\mathrm{r}\ell} S_{22}^\ell S_{21}^{\mathrm{r}\ell} \tag{4}$$

In (4), R^{r} (R^ℓ) is the reflection coefficient of the RHS (LHS) antenna as seen by the LHS (RHS) antenna.

By the nth orders-of-scattering approximation we mean an expansion of (3) in "powers" of R^{r} or R^ℓ up to and including order n [5]. It is important to realize that although we have treated the elements of the scattering-matrices as scalar numerical entities they are not. In general, the elements of the scattering-matrices are integral operators, and thus (2) contains not algebraic equations but rather integral equations in disguise. Therefore, the operators in the square brackets in (3) should be expanded in the Neumann series [6, §3.2],

$$[1 - G]^{-1} = 1 + G + GG + \cdots = \sum_{n=0}^{\infty} G^n, \quad ||G|| < 1, \tag{5}$$

rather than the Taylor series. In other words, the G^n term in (5) should be interpreted as the nth iterated kernel and 1 should be interpreted as the identity operator.

To gain some physical insight into (5) let's consider the 2nd orders-of-scattering approximation of S_{21} and S_{22}. In other words, expanding the square bracket terms in S_{21} and S_{22} in the Neumann series yields

$$S_{21}^{(2)} = S_{21}^{\mathrm{r}} S_{12}^{\mathrm{r}\ell} S_{21}^\ell + S_{21}^{\mathrm{r}} S_{12}^{\mathrm{r}\ell} \left(S_{22}^\ell R^{\mathrm{r}}\right) S_{21}^\ell + S_{21}^{\mathrm{r}} S_{12}^{\mathrm{r}\ell} \left(S_{22}^\ell R^{\mathrm{r}}\right)^2 S_{21}^\ell, \tag{6a}$$

and

$$S_{22}^{(2)} = S_{22}^{\mathrm{r}} + S_{21}^{\mathrm{r}} R^\ell S_{12}^{\mathrm{r}} + S_{21}^{\mathrm{r}} R^\ell \left(S_{11}^{\mathrm{r}} R^\ell\right) S_{12}^{\mathrm{r}}, \tag{6b}$$

where the superscript (2) denotes the order of the approximation. In Fig. 3, the first, second, and third terms on the RHS of (6a) are schematically shown by the solid, dashed, and dotted lines, respectively. From Fig. 3, we see that the first term corresponds to the direct transmission of a_1^ℓ and the second term includes the reflections of the signal by the probe and the AUT. A similar interpretation may be constructed for (6b), see Fig. 4. It is important to note that the first term on the RHS of (6b) corresponds to the direct reflection of a_2^{r} by the RHS antenna (probe) as seen from its feed. In other words, this is the reflection coefficient that one would measure in the absence of the LHS antenna (AUT).

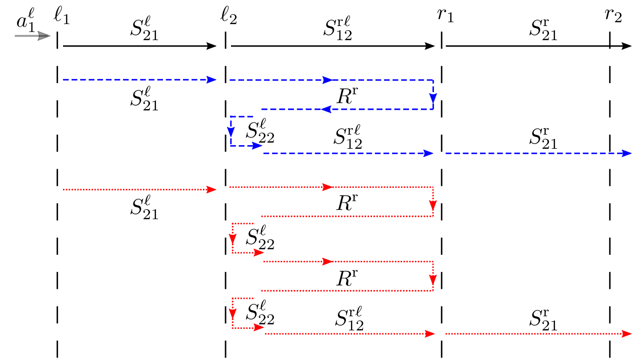

Fig. 3. (Color online) The 2nd orders-of-scattering approximation of $S_{21}^{(2)}$ is schematically illustrated.

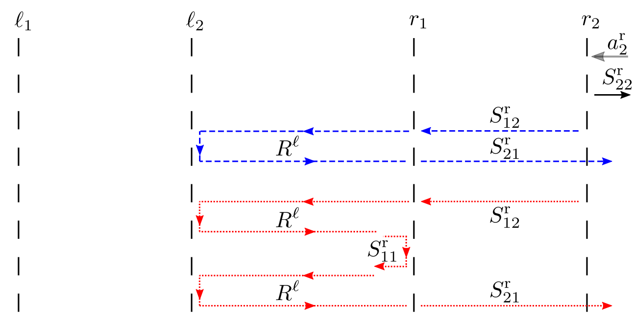

Fig. 4. (Color online) The 2nd orders-of-scattering approximation of $S_{22}^{(2)}$ is schematically illustrated.

III. Wacker's Equation

In the previous section, we provided a physical insight into the mutual coupling between the two antennas. This was done in a general and abstract manner by formal manipulation of the scattering-matrices. In this section, we will remove the abstraction layer by constructing an explicit form of the S-matrix.

If we assume the probe antenna is connected to a load with the reflection coefficient Γ^{r}, then $a_2^{\mathrm{r}} = \Gamma^{\mathrm{r}} b_2^{\mathrm{r}}$ and from (2) we have

$$b_2^{\mathrm{r}} = \left[1 - S_{22}\Gamma^{\mathrm{r}}\right]^{-1} S_{21} a_1^{\ell}. \tag{7}$$

Notice that the square bracket on the RHS of (7) contains the integral operator S_{22} and *not* the reflection coefficient of the probe S_{22}^{r}.

A. Zeroth Order-of-Scattering

To obtain the zeroth order-of-scattering approximation of (7) we substitute the first term on the RHS of (6) into (7) to obtain

$$b_2^{\mathrm{r}(0)} = \left[1 - S_{22}^{\mathrm{r}}\Gamma^{\mathrm{r}}\right]^{-1} S_{21}^{\mathrm{r}} S_{12}^{\mathrm{r}\ell} S_{21}^{\ell} a_1^{\ell}. \tag{8}$$

In (8), the term inside the square bracket is a complex *scalar* quantity, and thus we only need to determine the explicit form of

$$T \equiv S_{21}^{\mathrm{r}} S_{12}^{\mathrm{r}\ell} S_{21}^{\ell} a_1^{\ell}. \tag{9}$$

The key to obtaining the explicit form of T is to recognize that it may be written as [4, pp.12–13]

$$T = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_1^{\ell} f_1(k_x) f_2(k_y) f_3(k_z) \frac{\mathrm{e}^{+\mathrm{i}k_z d}}{k_z}\,\mathrm{d}k_x \mathrm{d}k_y, \tag{10a}$$

where f_1, f_2, and f_3 are *entire* functions and wave vector $\boldsymbol{k} = k_x\hat{\boldsymbol{e}}_{x^\ell} + k_y\hat{\boldsymbol{e}}_{y^\ell} + k_z\hat{\boldsymbol{e}}_{z^\ell}$ is such that

$$k_z = \begin{cases} +\sqrt{k^2-\kappa^2} & \text{for} \quad \kappa < k \\ +\mathrm{i}\sqrt{\kappa^2-k^2} & \text{for} \quad \kappa > k \end{cases}, \tag{10b}$$

with $\kappa^2 = k_x^2 + k_y^2$, see Fig. 2. Recall that for an entire function the principal part of the Laurent series vanishes [7, p.17], and thus we have

$$f_1 = \sum_{m=0}^{\infty} A_m^x k_x^m, \quad f_2 = \sum_{n=0}^{\infty} A_n^y k_y^n, \quad f_3 = \sum_{p=0}^{\infty} A_p^z k_z^p. \tag{11}$$

Substituting (11) into (10a), and noting that the odd powers of k_x and k_y integrate to zero, we obtain

$$T = a_1^{\ell} \sum_{mnp} A_m^x A_n^y A_p^z \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_x^{2m} k_y^{2n} k_z^p \frac{\mathrm{e}^{+\mathrm{i}k_z d}}{k_z}\,\mathrm{d}k_x\mathrm{d}k_y, \tag{12}$$

where the triple sum is over all non-negative integers. Converting the double integral in (12) to polar coordinates via $k_x = \kappa\cos\theta$ and $k_y = \kappa\sin\theta$, then using the orthogonal property of cosines and sines yields

$$T = a_1^{\ell} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} A_n^x A_n^y A_p^z \int_0^{\infty} \kappa^{4n} k_z^p \frac{\mathrm{e}^{+\mathrm{i}k_z d}}{k_z} \kappa\,\mathrm{d}\kappa. \tag{13}$$

Changing the integration variable in (13) from κ to k_z by recalling that κ^2 is a function of k_z^2, see (10b), yields

$$T = -a_1^{\ell} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} A_n^x A_n^y A_p^z \times \int_k^{\mathrm{i}\infty} \left(k^2 - k_z^2\right)^{2n} k_z^p \mathrm{e}^{+\mathrm{i}k_z d}\,\mathrm{d}k_z. \tag{14}$$

If we parameterize the line integral in the complex k_z-plane in (14) via $k_z = k + \mathrm{i}t$, then the product of the two infinite sums in (14) can be written as

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} A_n^x A_n^y A_p^z \left(k^2 - k_z^2\right)^{2n} k_z^p = \sum_{q=0}^{\infty} B_q t^q, \tag{15}$$

where the unknown B_q coefficients depend on the A_n^x, A_n^y, A_p^z coefficients and the wavenumber k. Substituting (15) into (14) and changing the integration variable from k_z to t yields

$$T = -\mathrm{i}a_1^{\ell} \sum_{q=0}^{\infty} B_q \mathrm{e}^{+\mathrm{i}kd} \int_0^{\infty} t^q \mathrm{e}^{-td}\,\mathrm{d}t. \tag{16}$$

Recognizing the integral in (16) as the Laplace transform of t^q [8, §17.13] we immediately obtain

$$T = a_1^{\ell} \frac{\mathrm{e}^{+\mathrm{i}kd}}{kd} \sum_{n=0}^{\infty} \frac{A_n}{(kd)^n}, \tag{17}$$

where the unknown A_n coefficients depend on the B_q coefficients. The form of (17) is not surprising because according to the Wilcox expansion theorem [9] any electromagnetic wave produced by a *finite* charge distribution has an expansion of the form

$$\frac{\mathrm{e}^{+\mathrm{i}kr}}{kr} \sum_{n=0}^{\infty} \frac{\boldsymbol{A}_n(\theta,\phi)}{(kr)^n}, \qquad r > r_c, \tag{18}$$

where (r, θ, ϕ) are the usual spherical coordinates and r_c is the radius of the smallest circumscribing sphere containing the finite charge distribution. Finally, after substituting (17) into (8) we obtain the zeroth order-of-scattering approximation of the signal measured by the probe; namely,

$$b_2^{\mathrm{r}(0)} = a_1^{\ell} \left[1 - S_{22}^{\mathrm{r}}\Gamma^{\mathrm{r}}\right]^{-1} \frac{\mathrm{e}^{+\mathrm{i}kd}}{kd} \sum_{n=0}^{\infty} \frac{A_n}{(kd)^n}. \tag{19}$$

From the derivation of (17) we see that the A_n coefficients in (19) are *independent* of S_{22}^{r} and Γ^{r}. This observation is obvious in the current context but it will become camouflaged when we consider all orders-of-scattering.

B. All Orders-of-Scattering

The first orders-of-scattering approximation of S_{21} includes the first two terms on the RHS of (6a); namely, the zeroth order-of-scattering, which we computed in Section III-A, and the $S_{21}^{\mathrm{r}} S_{12}^{\mathrm{r}\ell}\left(S_{22}^{\ell} R^{\mathrm{r}}\right) S_{21}^{\ell}$ term. Similar to the computation of the zeroth order-of-scattering approximation, the key to obtaining the explicit form of $\mathbb{T} = S_{21}^{\mathrm{r}} S_{12}^{\mathrm{r}\ell} \left(S_{22}^{\ell} R^{\mathrm{r}}\right) S_{21}^{\ell} a_1^{\ell}$ is to recognize that it may be written as [4, p.24]

$$\mathbb{T} = a_1^{\ell} \iint\limits_{-\infty}^{\infty} \mathrm{d}k_x \mathrm{d}k_y \iint\limits_{-\infty}^{\infty} \mathrm{d}k_x' \mathrm{d}k_y' \iint\limits_{-\infty}^{\infty} \mathrm{d}k_x'' \mathrm{d}k_y'' \; f(k_x, k_y, k_x', k_y', k_x'', k_y'') \frac{\mathrm{e}^{\mathrm{i}(k_z + k_z' + k_z'')d}}{k_z + k_z' + k_z''}, \tag{20}$$

where the function f is an entire function in each variable. The integrals in (20) may be evaluated following the procedure of Section III-A to obtain

$$\mathbb{T} = a_1^{\ell} \frac{\mathrm{e}^{3\mathrm{i}kd}}{(kd)^3} \sum_{n=0}^{\infty} \frac{A_n'}{(kd)^n}. \tag{21}$$

In general, we can use the above method for all orders-of-scattering. Therefore, we can obtain *exact* explicit forms of S_{21} and S_{22}. These forms are given by [4, pp.25–27],

$$S_{21} = \frac{\mathrm{e}^{\mathrm{i}kd}}{kd} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2m\mathrm{i}kd}}{(kd)^{2m}} \frac{A_{mn}}{(kd)^n}. \tag{22a}$$

and

$$S_{22} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2m\mathrm{i}kd}}{(kd)^{2m}} \frac{B_{mn}}{(kd)^n}, \tag{22b}$$

where $B_{0n} = S_{22}^{\mathrm{r}} \delta_{0n}$ and δ_{0n} is the Kronecker delta function. In (22b), $B_{0n} = S_{22}^{\mathrm{r}} \delta_{0n}$ because (22b) must agree with the zeroth order-of-scattering approximation, i.e., $S_{22}^{(0)} = S_{22}^{\mathrm{r}}$.

To obtain Wacker's equation, we use the Neumann series to expand the square bracket term in (7) and then substitute (22) into the resultant, i.e.,

$$\frac{b_2^{\mathrm{r}}}{a_1^{\ell}} = \frac{\mathrm{e}^{\mathrm{i}kd}}{kd} \times \sum_{p=0}^{\infty} \left[\Gamma^{\mathrm{r}} \sum_{mn} \frac{\mathrm{e}^{2m\mathrm{i}kd}}{(kd)^{2m}} \frac{B_{mn}}{(kd)^n}\right]^p \sum_{m'n'} \frac{\mathrm{e}^{2m'\mathrm{i}kd}}{(kd)^{2m'}} \frac{A_{m'n'}}{(kd)^{n'}} \tag{23}$$

Wacker's equation (23) may be written in a more traditional form by multiplying out the sums in (23) to obtain

$$\frac{b_2^{\mathrm{r}}}{a_1^{\ell}} = \frac{\mathrm{e}^{\mathrm{i}kd}}{kd} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2m\mathrm{i}kd}}{(kd)^{2m}} \frac{C_{mn}}{(kd)^n}, \tag{24}$$

where, in general, the C_{mn} coefficients depend on the A_{mn}, B_{mn} coefficients and Γ^{r}. We can simplify (24) further by noting that for $m = 0$ it must agree with the zeroth order-of-scattering approximation given by (19); thus, we have

$$\frac{b_2^{\mathrm{r}}}{a_1^{\ell}} = \frac{1}{1 - S_{22}^{\mathrm{r}} \Gamma^{\mathrm{r}}} \frac{\mathrm{e}^{\mathrm{i}kd}}{kd} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2m\mathrm{i}kd}}{(kd)^{2m}} \frac{F_{mn}}{(kd)^n}, \tag{25}$$

where F_{00} and only F_{00} is *independent* of S_{22}^{r} and Γ^{r} (see the end of Section III-A).

IV. Discussion

Wacker's equation is given by (25) and it accounts for *all* multiple scattering effects between the two antennas. Unfortunately, it can also be improperly derived leading to misconceptions. To see this, substitute the *exact* form of S_{21} given by (22a) into the zeroth order-of-scattering approximation (8) to obtain

$$\frac{1}{1 - S_{22}^{\mathrm{r}} \Gamma^{\mathrm{r}}} \frac{\mathrm{e}^{\mathrm{i}kd}}{kd} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2m\mathrm{i}kd}}{(kd)^{2m}} \frac{A_{mn}}{(kd)^n}. \tag{26}$$

This improper derivation suggests that Wacker's equation accounts for the multiple scattering effects contained in S_{22} only to the zeroth order-of-scattering. Of course, this conclusion is incorrect as we have shown in Section III-B.

Another source of confusion comes from the separation distance variable d. In general, d should be measured from behind the face of the AUT to behind the face of the probe as shown in Fig. 2. With this choice, the sums in (25) will usually converge for any d as long as the two antennas are not touching [4, pp.15–22]. A sufficient, but not necessary, condition for the sums in (25) to converge is given by

$$d > r_{\mathrm{AUT}} + r_{\mathrm{probe}}, \tag{27}$$

where r_{AUT} (r_{probe}) is the radius of the smallest circumscribing sphere containing the AUT (probe) [3, Ch. III, §5]. It is important to note that if the sums in (25) converge, then the on-axis gain and polarization of the antenna are *not* affected by the choice of origin of the separation distance. To see this, we shift the coordinate system by d_0 so that the new separation distance is given by

$$d' = d - d_0 \tag{28}$$

and substitute (28) into (25) to obtain

$$\frac{b_2^{\mathrm{r}}}{a_1^{\ell}} = \frac{1}{1 - S_{22}^{\mathrm{r}} \Gamma^{\mathrm{r}}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{(2m+1)\mathrm{i}kd'}}{(kd')^{2m+1+n}} \frac{F_{mn} \mathrm{e}^{(2m+1)\mathrm{i}kd_0}}{(1+\varepsilon)^{2m+n+1}}, \tag{29}$$

where $\varepsilon = d_0/d'$. Then, expanding $1/(1+\varepsilon)^{2m+n+1}$ in the binomial series and relabeling the expansion coefficients yields

$$\frac{b_2^{\mathrm{r}}}{a_1^{\ell}} = \frac{1}{1 - S_{22}^{\mathrm{r}} \Gamma^{\mathrm{r}}} \frac{\mathrm{e}^{\mathrm{i}kd'}}{kd'} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2m\mathrm{i}kd'}}{(kd')^{2m}} \frac{F_{mn}'}{(kd')^n}, \tag{30}$$

where $F_{00}' = \mathrm{e}^{\mathrm{i}kd_0} F_{00}$. In the far field, the gain and polarization are independent of the $F_{mn}', m > 0, n > 0$ coefficients. Thus, if we tactfully assume that the different polarization states of the antenna are measured in the same coordinate system, then from $F_{00}' = \mathrm{e}^{\mathrm{i}kd_0} F_{00}$ we see that the gain and polarization of the antenna are unchanged by an absolute phase shift [1, p.427]. In other words, only the relative phase between the different polarization states is of consequence.

V. Conclusions

In this paper, we provided orders-of-scattering interpretation of the scattering-matrix for a two antenna system. We obtained the orders-of-scattering interpretation by first expressing the scattering-matrix of the system in terms of its individual components before formally expanding it in the Neumann series, see Section II. Using the series representation of the scattering-matrix we obtained the explicit forms of its elements and derived Wacker's equation. We showed that Wacker's equation is exact and discussed its convergence properties. Furthermore, we also discussed from where the separation distance between the two antennas should be measured and showed that the choice of the origin does not affect the gain and the polarization of the antenna.

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