A 3-D Tensorial Integral Formulation of Scattering Containing Intriguing Relations

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***Abstract*— We investigate the role of the electric field and its normal derivative in 3-D electromagnetic scattering theory. We present an alternative integral equation formulation that uses the electric field and its normal derivative as the boundary unknowns. In particular, we extend a traditional formulation that is used in 2-D scattering theory to three-dimensions. We uncover several intriguing relationships involving closed surface integrals of the field and/or its derivative. In order not to obscure the physical/geometric awareness, the derivation is made from a tensor calculus perspective.**

***Index Terms*— Boundary conditions, boundary value problems, electrodynamics, integral equations, waves.**

I. Introduction

PREVIOUSLY DeSanto and Yuffa [1], [2] published a set of integral equations for electromagnetic (EM) scattering, where the electric field and its normal derivative were used to satisfy the boundary conditions. Unfortunately, the derivation presented in [1] is only valid in a Cartesian coordinate system and requires the scattering surface to be of the form $z = f(x, y)$. Furthermore, the formulation is presented in a vector component form, and thus it cannot be effortlessly extended to curvilinear coordinate systems. In addition, the employment of a nonunit normal vector used in the formulation is also troublesome from theoretical and numerical points of view. The net effect of the above-mentioned shortcomings is the loss of physical/geometrical insight. Arguably, this is the most detrimental sacrifice committed in our previous derivation. The main purpose of this paper is to remedy these shortcomings and to gain a physical insight into the mathematical relationships involving the electric field and its normal derivative.

Although this paper is theoretical in nature, we would like to remark that our formalism may be advantageous in some avant-garde applications. For decades, if not centuries, cloaking (invisibility) devices have fascinated minds. Up until now, these fascinating devices only existed in the realm of science fiction. Technological advances of this century brought these devices into the realm of reality. Recent experiments at microwave frequencies [3] and, to a lesser extent, at optical frequencies [4], [5] have demonstrated their feasibility. It is well-known that the boundary conditions have an extensive effect on the solution, and thus it is not surprising that unorthodox boundary conditions provide the most elegant means for achieving a solution. Indeed, for cloaking applications, such unorthodox boundary conditions on the normal components of the EM field or its derivative, rather than on the tangential components, provide the required solution. In particular, it has been shown that the vanishing of these normal components is required to achieve some cloaking aspects [6]–[14]. Although these boundary conditions were first considered by Rumsey [15] more than half a century ago, it is only in the past decade that they have started to attract attention in the literature. Therefore, the intriguing relationships derived in this paper, especially those involving the electric field and its normal derivative, as well as interrelationships between their normal/tangential components, may offer some insight into these fascinating devices.

Our formalism may also be advantageous in certain near field to far field (NF2FF) transformations that are commonly used in antenna metrology where the far field of an antenna is predicted from measurements in the near field. Traditionally, these NF2FF transformations were based on partial wave (modal) expansion of the field with the expansion coefficients computed from the near-field data taken on a canonical surface [16]–[18]. In the past few decades, NF2FF transformations based on integral equations started to appear in [19]–[23]. These equivalent current methods have a number of advantages with regard to the sampling requirements as well as the size and shape of the near-field measurement surface. It has been shown in [24] that a receiving (probe) antenna can be thought of as a linear differential operator that converts the incident field and its derivatives into an output voltage; hence, an integral equation method such as ours that inherently uses the field and its normal derivative may offer further physical insight. We plan to explore these ideas in future publications.

This paper is organized as follows. In Section II, we formulate the scattering problem and introduce the notation used in this paper. At this point, we also invite the reader to review tensor calculus (see Appendix A) because it is heavily used throughout this paper. In Section III, we derive the key continuity condition for the normal derivative of the electric field across an interface. Along the way, we also derive a number of intriguing relationships involving the normal derivative of the electric field. These are presented as corollaries in this section. In Section IV, we use the continuity condition derived in

Manuscript received August 25, 2017; revised June 4, 2018; accepted July 4, 2018. Date of publication July 12, 2018; date of current version October 4, 2018. This work was supported by the U.S. Government. *(Corresponding author: Alex J. Yuffa.)*

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Digital Object Identifier 10.1109/TAP.2018.2855671

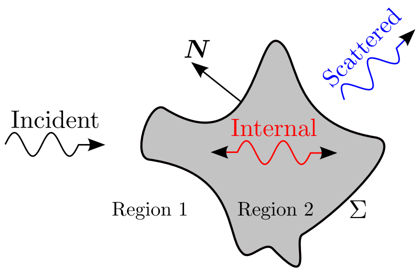

Fig. 1. Typical scattering geometry is shown, where the primary wave is incident from Region 1 onto a scatterer occupying Region 2. The scatterer is bounded by the surface Σ with the unit normal vector $\boldsymbol{N}$.

Section III to obtain the main result of this paper, i.e., the alternative surface integral equations. In Section V, we make a connection with the Stratton–Chu formula, and we conclude this paper in Section VI.

II. Problem Statement and Notation

Consider a typical scattering problem depicted in Fig. 1, where the incident electric field $\overset{\text{inc}}{\boldsymbol{E}}$ is assumed to satisfy the vector Helmholtz equation and the scattered field $\overset{\text{sca}}{\boldsymbol{E}}$ satisfies the Silver–Müller radiation condition [25, Sec. 4.2]. The scatterer (Region 2) is characterized by complex, constant permittivity, and permeability, i.e., $\overset{2}{\epsilon} \in \mathbb{C}$ and $\overset{2}{\mu} \in \mathbb{C}$, respectively. Region 1 is also characterized by a constant permittivity and permeability, but these constants are assumed to be purely real. Throughout this paper, we use the Gaussian unit system and assume that all fields are harmonic in time with a suppressed $\exp(-\mathrm{i}\omega t)$ time factor. This convention implies that the imaginary parts of $\overset{2}{\epsilon}$ and $\overset{2}{\mu}$ must be nonnegative [26]. Furthermore, we use tensor notation with the Einstein summation convention throughout this paper. In this notation, the Latin alphabet indices range from 1 to 3 and the Greek alphabet indices range from 1 to 2. We denote the coordinates of a source point in the 3-D Euclidean space by Z^1, Z^2, Z^3, or Z^i or simply by Z. Similarly, we denote the coordinates of an observation field point by $\widetilde{Z}^i$. If we let $\boldsymbol{R}$ denote the position vector, then the position vectors of a source point and a field point are given by $\boldsymbol{P} = \boldsymbol{R}(Z)$ and $\widetilde{\boldsymbol{P}} = \boldsymbol{R}(\widetilde{Z})$, respectively, and the covariant ambient basis is obtained from $\boldsymbol{R}$ via

$$\boldsymbol{Z}_i = \frac{\partial}{\partial Z^i}\boldsymbol{R}(Z). \tag{1}$$

If S^α (S^1 and S^2) denotes the surface coordinates, e.g., the coordinates of the surface Σ shown in Fig. 1, then the surface covariant basis is given by

$$\boldsymbol{S}_\alpha = \frac{\partial}{\partial S^\alpha}\boldsymbol{R}(Z(S)) = Z^i_\alpha \boldsymbol{Z}_i \tag{2}$$

where $Z^i_\alpha = \partial Z^i/\partial S^\alpha$ is the shift tensor. Furthermore, the covariant metric tensor and the surface covariant metric tensor are given by $Z_{ij} = \boldsymbol{Z}_i \cdot \boldsymbol{Z}_j$ and $S_{\alpha\beta} = \boldsymbol{S}_\alpha \cdot \boldsymbol{S}_\beta$, respectively.

To obtain an integral representation of the E-field in Region # (# denotes 1 or 2), we multiply $(\nabla_i\nabla^i + \overset{\#}{k}{}^2)\overset{\#}{\boldsymbol{E}} = \boldsymbol{0}$ by $\overset{\#}{G}$ and $(\nabla_i\nabla^i + \overset{\#}{k}{}^2)\overset{\#}{G} = -\delta(\widetilde{\boldsymbol{P}} - \boldsymbol{P})$ by $\overset{\#}{\boldsymbol{E}}$, then take the difference between the two equations. After integrating the resultant equation over Region # and using Gauss's theorem as well as the sifting property of the Dirac delta function $\delta(\widetilde{\boldsymbol{P}} - \boldsymbol{P})$, we obtain

$$\overset{\text{inc}}{\boldsymbol{E}}(\widetilde{Z}) - \int_\Sigma \left[\overset{1}{G}(\widetilde{Z}, S)\frac{\partial}{\partial N}\overset{1}{\boldsymbol{E}}(S) - \overset{1}{\boldsymbol{E}}(S)\frac{\partial}{\partial N}\overset{1}{G}(\widetilde{Z}, S)\right]\mathrm{d}S = \begin{cases}\overset{1}{\boldsymbol{E}}(\widetilde{Z}), & \widetilde{Z} \in \text{Region 1}\\ 0, & \widetilde{Z} \in \text{Region 2}\end{cases} \tag{3a}$$

and

$$\int_\Sigma \left[\overset{2}{G}(\widetilde{Z}, S)\frac{\partial}{\partial N}\overset{2}{\boldsymbol{E}}(S) - \overset{2}{\boldsymbol{E}}(S)\frac{\partial}{\partial N}\overset{2}{G}(\widetilde{Z}, S)\right]\mathrm{d}S = \begin{cases}\overset{2}{\boldsymbol{E}}(\widetilde{Z}), & \widetilde{Z} \in \text{Region 2}\\ 0, & \widetilde{Z} \in \text{Region 1}\end{cases} \tag{3b}$$

where $\overset{\#}{\boldsymbol{E}}$ is the total E-field in Region #, $\overset{\#}{k}$ is the wavenumber in Region #, and $\overset{\#}{G}$ denotes the free-space Green's function in Region #. Explicitly, the free-space Green's function is given by

$$\overset{\#}{G}(\widetilde{\boldsymbol{P}}, \boldsymbol{P}) = \frac{\exp\left(\mathrm{i}\overset{\#}{k}\,\|\widetilde{\boldsymbol{P}} - \boldsymbol{P}\|\right)}{4\pi\,\|\widetilde{\boldsymbol{P}} - \boldsymbol{P}\|}. \tag{4}$$

In (3), $\partial/\partial N$ denotes the normal derivative with respect to the source coordinates, that is,

$$\frac{\partial}{\partial N} = N^i\nabla_i \tag{5}$$

where the unit normal $\boldsymbol{N} = N^i\boldsymbol{Z}_i$ points from Region 2 into Region 1 and ∇_i denotes the covariant derivative. Furthermore, in the derivation of (3), we expressed the Laplacian $\nabla_i\nabla^i$ in terms of the covariant and contravariant derivatives.

If the E-field and its normal derivative are known on Σ, then we can compute the E-field everywhere in space via (3). In order to find the E-field and its normal derivative on Σ, we let the field point $\widetilde{Z}$ approach the surface and note that the Green's function and its normal derivative are singular when $Z = \widetilde{Z}$. The Green's function singularity contributes 0 to the integral and its normal derivative contributes $+1/2$ ($-1/2$) to the integral if the surface is approached from Region 1 (2) [27, Sec. 3.1.1]. Thus, taking the limit as $\widetilde{Z}$ approaches the surface yields

$$\overset{\text{inc}}{\boldsymbol{E}}(\widetilde{S}) - ⨍_\Sigma \left[\overset{1}{G}\frac{\partial\overset{1}{\boldsymbol{E}}}{\partial N} - \overset{1}{\boldsymbol{E}}\frac{\partial\overset{1}{G}}{\partial N}\right]\mathrm{d}S = \frac{1}{2}\overset{1}{\boldsymbol{E}}(\widetilde{S}) \tag{6a}$$

and

$$⨍_\Sigma \left[\overset{2}{G}\frac{\partial\overset{2}{\boldsymbol{E}}}{\partial N} - \overset{2}{\boldsymbol{E}}\frac{\partial\overset{2}{G}}{\partial N}\right]\mathrm{d}S = \frac{1}{2}\overset{2}{\boldsymbol{E}}(\widetilde{S}) \tag{6b}$$

where $⨍$ denotes the Cauchy principal value integral and the functional arguments of the integrands have been omitted to conserve space. Having chosen the E-field and its normal derivative as the boundary unknowns, we notice that (6) has twice as many unknowns as equations. A natural approach to reduce the number of unknowns in (6) is to supplement it with continuity conditions that relate the E-field and its normal derivative across the surface Σ. To derive these conditions, we have to rely on tensor calculus. Although there is a plethora of textbooks on tensor calculus there seems to be no uniformity in notation and, to borrow a phrase from renowned Russian physicist Lev D. Landau, "the theoretical minimum" differs greatly from book to book. To remedy this situation and to make this paper self-contained as much as possible, we have included a brief tutorial on tensor calculus in Appendix A. This tutorial is similar in flavor to [28] and, to a lesser extent, to [29].

Finally, we remark the assumption that the incident field satisfies the vector Helmholtz equation can be relaxed without significantly altering our method. To see this, simply use the scattered field $\overset{\text{sca}}{\boldsymbol{E}}$ instead of $\overset{1}{\boldsymbol{E}}$ in the derivation of (3a). This alternative derivation will yield (3a) with an additional (known) term on the left-hand side (LHS), namely,

$$\int_\Sigma \left[\overset{1}{G}(\widetilde{Z}, S)\frac{\partial}{\partial N}\overset{\text{inc}}{\boldsymbol{E}}(S) - \overset{\text{inc}}{\boldsymbol{E}}(S)\frac{\partial}{\partial N}\overset{1}{G}(\widetilde{Z}, S)\right] \mathrm{d}S.$$

III. Continuity Conditions

We will derive the continuity conditions for the E-field and its normal derivative directly from the Maxwell equations

$$\nabla \times \boldsymbol{E} - \mathrm{i}\frac{\mu\omega}{c}\boldsymbol{H} = \boldsymbol{0} \tag{7a}$$

$$\nabla \times \boldsymbol{H} + \mathrm{i}\frac{\epsilon\omega}{c}\boldsymbol{E} = \boldsymbol{0} \tag{7b}$$

$$\nabla \cdot \begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{Bmatrix} = 0 \tag{7c}$$

and the conventional continuity conditions, namely,

$$\boldsymbol{N} \times \begin{Bmatrix} \overset{1}{\boldsymbol{E}} - \overset{2}{\boldsymbol{E}} \\ \overset{1}{\boldsymbol{H}} - \overset{2}{\boldsymbol{H}} \end{Bmatrix} = \boldsymbol{0} \tag{8a}$$

and

$$\boldsymbol{N} \cdot \begin{Bmatrix} \overset{1}{\epsilon}\overset{1}{\boldsymbol{E}} - \overset{2}{\epsilon}\overset{2}{\boldsymbol{E}} \\ \overset{1}{\mu}\overset{1}{\boldsymbol{H}} - \overset{2}{\mu}\overset{2}{\boldsymbol{H}} \end{Bmatrix} = 0. \tag{8b}$$

By noting that (8a) and (8b) specify the continuity conditions on the tangential and normal components of the field, respectively, we can immediately rewrite (8) as [also see (A.1)]

$$\overset{2}{\boldsymbol{E}} = \overset{\text{r}}{\epsilon}^{-1}(\boldsymbol{N} \cdot \overset{1}{\boldsymbol{E}})\boldsymbol{N} + (\boldsymbol{S}^\alpha \cdot \overset{1}{\boldsymbol{E}})\boldsymbol{S}_\alpha \tag{9}$$

where $\overset{\text{r}}{\epsilon} = \overset{2}{\epsilon}/\overset{1}{\epsilon}$. Using the projection decomposition formula (A.4), we can express the tangential projection on the right-hand side (RHS) of (9) in terms of the normal projection to obtain

$$\overset{2}{E}{}^i = A^i_j \overset{1}{E}{}^j, \quad \text{where} \quad A^i_j = \delta^i_j + (\overset{\text{r}}{\epsilon}^{-1} - 1)N^i N_j \tag{10}$$

and δ^i_j is the Kronecker symbol. We prefer to work with (10) instead of (9) because, from a computational point of view, it is more convenient to work with unit normals than shift tensors. Regardless how one chooses to express A^i_j, it is important to note that each component of the E-field in one region depends on all of the components of the E-field in the other region. In other words, in general A^i_j does not have any zero entries.

A. Normal Derivative Continuity

To derive a continuity condition for the normal derivative of the E-field, we first express the normal derivative in the form

$$\frac{\partial \boldsymbol{E}}{\partial N} = \nabla_\alpha \boldsymbol{U}^\alpha + \boldsymbol{V} \tag{11}$$

where $\boldsymbol{U}^\alpha$ may depend on the E-field but not on its derivatives. This form is desired because we can apply Gauss's theorem (A.13) to the first term on the RHS of (11) and obtain terms that do not depend on the derivatives of the E-field. In other words, after the application of Gauss's theorem, we will be able to use (10) to express the RHS of (11) in terms of the E-field in the other region.

We begin the derivation by expanding $\boldsymbol{N} \times (\nabla \times \boldsymbol{E})$ in terms of two inner products and, after using (7a), we obtain

$$N^m \nabla_m E_\ell = N^m \nabla_\ell E_m + \mathrm{i}\frac{\mu\omega}{c} K_\ell \tag{12}$$

where $\boldsymbol{K} = -\boldsymbol{N} \times \boldsymbol{H}$. The term on the LHS is the desired normal derivative, but the first term on the RHS contains ∇_ℓ instead of the desired ∇_α. These two covariant derivatives are related by the chain rule [28, Sec. 11.8], namely,

$$\nabla_\alpha E_m = Z^k_\alpha \nabla_k E_m. \tag{13}$$

It is important to have a proper interpretation of the above-mentioned elegant but terse relationship. On the LHS, we have the covariant surface derivative of the *surface restriction* of the ambient field, but on the RHS, we have the projection (shift tensor Z^k_α) of the *ambient* covariant derivative. Multiplying (13) by Z^β_ℓ and then contracting the Greek indices yields

$$\nabla_\ell E_m = Z^\alpha_\ell \nabla_\alpha E_m + N_\ell \frac{\partial E_m}{\partial N} \tag{14}$$

where we used (A.4) to express the tangential projection operator $Z^\alpha_\ell Z^k_\alpha$ in terms of the normal projection operator $N_\ell N^k$. We can loosely interpret (14) as the "reverse" chain rule after moving the last term on the RHS to the LHS and comparing the resultant RHS with the RHS of (13). Substituting (14) into the RHS of (12) yields

$$\frac{\partial E_\ell}{\partial N} = N^m Z^\alpha_\ell \nabla_\alpha E_m + N_\ell N^m \frac{\partial E_m}{\partial N} + \mathrm{i}\frac{\mu\omega}{c} K_\ell. \tag{15}$$

The second term on the RHS of (15) contains the normal component of the normal derivative of the E-field, i.e., $N^m \partial E_m/\partial N$. This component is related to the surface covariant derivative of the E-field because the E-field is divergenceless in source-free space, i.e., $\nabla \cdot \boldsymbol{E} = 0$. Thus, raising the ℓ index in (14) and contracting it with the m index yields

$$N^m \frac{\partial E_m}{\partial N} = -Z_{m\alpha} \nabla^\alpha E^m. \tag{16}$$

Substituting (16) into (15) and using the product rule, we obtain

$$\frac{\partial E_\ell}{\partial N} = \nabla^\alpha (N_m E^m Z_{\ell\alpha} - Z_{m\alpha} E^m N_\ell) + \mathrm{i}\frac{\mu\omega}{c} K_\ell + E^m \nabla^\alpha (N_\ell Z_{m\alpha} - N_m Z_{\ell\alpha}). \tag{17}$$

This would be the desired relationship if the last line on the RHS vanished. Fortunately it does, and we prove this in Appendix B. Thus, after using the metrinilic properties of ∇_i and ∇_α (see Sections B and C in Appendix A), we finally have the desired relationship

$$\frac{\partial \boldsymbol{E}}{\partial N} = \nabla^\alpha[(\boldsymbol{N}\cdot\boldsymbol{E})\boldsymbol{S}_\alpha - (\boldsymbol{S}_\alpha\cdot\boldsymbol{E})\boldsymbol{N}] + \mathrm{i}\frac{\mu\omega}{c}\boldsymbol{K}. \quad (18)$$

The two terms inside the square brackets on the RHS are unusual because the surface tangent vector $\boldsymbol{S}_\alpha$ is scaled by the *normal* component of the E-field $\boldsymbol{N}\cdot\boldsymbol{E}$, and the normal vector $\boldsymbol{N}$ is scaled by the *tangential* components of the E-field $\boldsymbol{S}_\alpha\cdot\boldsymbol{E}$. Nonetheless, we can use (18) to derive the continuity condition for the normal derivative of the E-field. To see this, write (18) in Regions 1 and 2 and then consider $\overset{1}{\mu}(\partial/\partial N)\overset{2}{\boldsymbol{E}} - \overset{2}{\mu}(\partial/\partial N)\overset{1}{\boldsymbol{E}}$ to obtain

$$\frac{\partial \overset{2}{\boldsymbol{E}}}{\partial N} = \overset{r}{\mu}\left(\frac{\partial \overset{1}{\boldsymbol{E}}}{\partial N} - \nabla^\alpha[(\boldsymbol{N}\cdot\overset{1}{\boldsymbol{E}})\boldsymbol{S}_\alpha - (\boldsymbol{S}_\alpha\cdot\overset{1}{\boldsymbol{E}})\boldsymbol{N}]\right) + \nabla^\alpha[(\boldsymbol{N}\cdot\overset{2}{\boldsymbol{E}})\boldsymbol{S}_\alpha - (\boldsymbol{S}_\alpha\cdot\overset{2}{\boldsymbol{E}})\boldsymbol{N}] \quad (19)$$

where $\overset{r}{\mu} = \overset{2}{\mu}/\overset{1}{\mu}$. The surface currents do not appear in (19) because $\boldsymbol{K}$ is continuous across an interface (see (8a)). The continuity condition given by (19) is the main result of this section; it relates the normal derivative of the E-field across an interface. To gain some insight into (19), it is instructive to consider (19) under special circumstances. In practice, we often deal with nonmagnetic media ($\overset{r}{\mu} = 1$), and in this case (19) reduces to

$$\frac{\partial \overset{2}{\boldsymbol{E}}}{\partial N} - \nabla^\alpha[(\boldsymbol{N}\cdot\overset{2}{\boldsymbol{E}})\boldsymbol{S}_\alpha] = \frac{\partial \overset{1}{\boldsymbol{E}}}{\partial N} - \nabla^\alpha[(\boldsymbol{N}\cdot\overset{1}{\boldsymbol{E}})\boldsymbol{S}_\alpha]. \quad (20)$$

From (20), we see that the discontinuity of the normal derivative of the E-field is dictated by the normal component of the E-field, i.e., by $\boldsymbol{N}\cdot\boldsymbol{E}$. Another situation of practical importance is when a 3-D scattering problem can be treated as a 2-D one. Here, we will assume that all fields and the scattering surface are independent of the z-coordinate and the incident wave is polarized along the z-axis (TM mode). Under these conditions, $\boldsymbol{N}\cdot\boldsymbol{E} = 0$ and, assuming $\boldsymbol{S}_2$ is a constant vector parallel to the z-axis, $\boldsymbol{S}_1\cdot\boldsymbol{E} = 0$. Furthermore, $\nabla^2(\boldsymbol{S}_2\cdot\boldsymbol{E})\boldsymbol{N} = 0$ because both the field and the surface are independent of z. Therefore, (19) reduces to the well-known continuity condition [30, Sec. 14.1], that is,

$$\frac{1}{\overset{2}{\mu}}\frac{\partial \overset{2}{\boldsymbol{E}}}{\partial N} = \frac{1}{\overset{1}{\mu}}\frac{\partial \overset{1}{\boldsymbol{E}}}{\partial N}. \quad (21)$$

B. Normal Derivative Corollaries

From the relationships derived in Sec. III-A, we can derive four elegant and intriguing formulas. The first formula we shall derive relates the surface integral of the normal component of $\partial\boldsymbol{E}/\partial N$ to the surface integral of $\boldsymbol{N}\cdot\boldsymbol{E}$ and the mean curvature. (For the reader's convenience, the mean curvature and related terms are discussed in Appendix A.) To derive this formula, we use the product rule to rewrite (16) as

$$N_m\frac{\partial E^m}{\partial N} = -\nabla^\alpha(Z_\alpha^m E_m) + E_m\nabla^\alpha Z_\alpha^m \quad (22)$$

and then relate $\nabla^\alpha Z_\alpha^m$ to the curvature tensor via (A.11). This yields

$$\boldsymbol{N}\cdot\frac{\partial \boldsymbol{E}}{\partial N} = (\boldsymbol{N}\cdot\boldsymbol{E})\,W_\alpha^\alpha - \nabla^\alpha(\boldsymbol{S}_\alpha\cdot\boldsymbol{E}) \quad (23)$$

where W_β^α is the curvature tensor. Integrating (23) over an arbitrary closed surface Σ and using Gauss's theorem (A.13) on the last term on the RHS yields

$$\int_\Sigma \boldsymbol{N}\cdot\frac{\partial \boldsymbol{E}}{\partial N}\mathrm{d}S = \int_\Sigma(\boldsymbol{N}\cdot\boldsymbol{E})\,W_\alpha^\alpha\mathrm{d}S - \int_{\partial\Sigma} n^\alpha(\boldsymbol{S}_\alpha\cdot\boldsymbol{E})\,\mathrm{d}C$$

where n^α is given by (A.12). The last term on the RHS vanishes because the integral is over the nonexistent boundary of the closed surface Σ (see Section D in Appendix A), and thus we obtain the desired formula

$$\int_\Sigma \boldsymbol{N}\cdot\frac{\partial \boldsymbol{E}}{\partial N}\,\mathrm{d}S = \int_\Sigma(\boldsymbol{N}\cdot\boldsymbol{E})\,W_\alpha^\alpha\,\mathrm{d}S. \quad (24)$$

From (24), we immediately have that

$$\int_\Sigma \boldsymbol{N}\cdot\frac{\partial \boldsymbol{E}}{\partial N}\,\mathrm{d}S = 0$$

for closed surfaces with a zero mean curvature, i.e., $W_\alpha^\alpha = 0$. These surfaces are usually called *minimal surfaces* [31, Ch. 2] and are of current scientific interest in mathematics, physics, and computer graphics.

We can also derive a formula analogous to (24) for the tangential components of $\partial\boldsymbol{E}/\partial N$. To see this, substitute (18) into $\boldsymbol{S}^\beta\cdot(\partial\boldsymbol{E}/\partial N)$ and use (A.10) to obtain

$$\boldsymbol{S}^\beta\cdot\frac{\partial \boldsymbol{E}}{\partial N} = \nabla^\alpha\left[(\boldsymbol{N}\cdot\boldsymbol{E})\,\boldsymbol{S}^\beta\cdot\boldsymbol{S}_\alpha\right] + (\boldsymbol{S}_\alpha\cdot\boldsymbol{E})\,W^{\alpha\beta} + \mathrm{i}\frac{\mu\omega}{c}\boldsymbol{S}^\beta\cdot\boldsymbol{K}. \quad (25)$$

Integrating (25) over an arbitrary closed surface Σ and note that the first term on the RHS vanishes by Gauss's theorem (A.13) yields

$$\int_\Sigma \boldsymbol{S}^\beta\cdot\frac{\partial \boldsymbol{E}}{\partial N}\,\mathrm{d}S = \int_\Sigma(\boldsymbol{S}_\alpha\cdot\boldsymbol{E})\,W^{\alpha\beta}\,\mathrm{d}S - \mathrm{i}\frac{\mu\omega}{c}\int_\Sigma \varepsilon^{\alpha\beta}(\boldsymbol{S}_\alpha\cdot\boldsymbol{H})\,\mathrm{d}S \quad (26)$$

where $\varepsilon^{\alpha\beta}$ denotes the Levi–Civita symbol. From (26), we see that the surface integral of the tangential components of $\partial\boldsymbol{E}/\partial N$ depends on the tangential components of the E-field as well as the H-field. This is in contrast with the normal component of $\partial\boldsymbol{E}/\partial N$ that depends only on the E-field (see (24)).

We can also derive a relationship involving all components of the normal derivative of the E-field. Applying Gauss's theorem in the same manner as used earlier to (18) yields

$$\int_\Sigma \frac{\partial \boldsymbol{E}}{\partial N}\,\mathrm{d}S = -\mathrm{i}\frac{\mu\omega}{c}\int_\Sigma \boldsymbol{N}\times\boldsymbol{H}\,\mathrm{d}S. \quad (27)$$

From (27), we see that the surface integral of $\partial\boldsymbol{E}/\partial N$ depends only on the tangential components of the H-field. The formulas for the surface integral of $\partial\boldsymbol{E}/\partial N$ and its normal/tangential components are intriguing. They provide some insight into how the normal derivative depends on the normal and/or tangential components of the EM fields. Furthermore, these formulas and the following equation may be used computationally to gauge the accuracy of the numerically computed boundary unknowns.

The last formula we should derive is analogous to (21) but holds for *any* sufficiently smooth homogeneous scatterer. Applying our Gauss's theorem argument to the continuity condition (19) yields

$$\frac{1}{\overset{2}{\mu}}\int_{\Sigma}\frac{\partial \overset{2}{\boldsymbol{E}}}{\partial N}\,\mathrm{d}S=\frac{1}{\overset{1}{\mu}}\int_{\Sigma}\frac{\partial \overset{1}{\boldsymbol{E}}}{\partial N}\,\mathrm{d}S \tag{28}$$

where Σ is the surface of the scatterer and not an arbitrary closed surface. This remarkable relationship states that the normal derivative of the E-field is continuous across an interface in a weighted mean sense. Of course, from (21), we see that if the scattering problem is 2-D, then the continuity holds in a pointwise sense as well.

IV. Alternative Integral Equations

The continuity conditions for the electric field and its normal derivative are given by (10) and (19), respectively. Equipped with these conditions, we can readily derive a set of surface integral equations with $\boldsymbol{E}$ and $\partial\boldsymbol{E}/\partial N$ as the boundary unknowns. If we are predominantly interested in the E-field in Region 1, then (3a) suggests that we choose $\overset{1}{\boldsymbol{E}}$ and $\partial\overset{1}{\boldsymbol{E}}/\partial N$ as the boundary unknowns. On the other hand, if we are more interested in the internal field, i.e., the E-field in Region 2, then (3b) suggests that we choose $\overset{2}{\boldsymbol{E}}$ and $\partial\overset{2}{\boldsymbol{E}}/\partial N$ as the boundary unknowns. For concreteness we will choose the former, but remark that for the latter choice the derivation is analogous.

Substituting (19) into (6b) and using the product rule yields

$$\frac{1}{2}\overset{2}{\boldsymbol{E}}(\widetilde{S})=\fint_{\Sigma}\left[\overset{r}{\mu}\overset{2}{G}\frac{\partial\overset{1}{\boldsymbol{E}}}{\partial N}-\overset{2}{\boldsymbol{E}}\frac{\partial\overset{2}{G}}{\partial N}+\boldsymbol{F}_{\alpha}\nabla^{\alpha}\overset{2}{G}\right]\mathrm{d}S-\fint_{\Sigma}\nabla^{\alpha}(\boldsymbol{F}_{\alpha}\overset{2}{G})\mathrm{d}S \tag{29a}$$

where

$$\boldsymbol{F}_{\alpha}=\boldsymbol{N}\cdot(\overset{r}{\mu}\overset{1}{\boldsymbol{E}}-\overset{2}{\boldsymbol{E}})\boldsymbol{S}_{\alpha}-\boldsymbol{S}_{\alpha}\cdot(\overset{r}{\mu}\overset{1}{\boldsymbol{E}}-\overset{2}{\boldsymbol{E}})\boldsymbol{N}. \tag{29b}$$

The last term on the RHS of (29a) vanishes by the same Gauss's theorem argument that was used to reduce (23) to (24). The third term on the RHS of (29a) can be written in terms of the gradient $\nabla\equiv\boldsymbol{Z}^{i}\nabla_{i}$, and the normal derivative of Green's function by using the self-evident identity

$$\boldsymbol{Z}^{i}\nabla_{i}\overset{2}{G}=\boldsymbol{S}_{\alpha}\nabla^{\alpha}\overset{2}{G}+\boldsymbol{N}\frac{\partial\overset{2}{G}}{\partial N} \tag{30}$$

to eliminate $\boldsymbol{S}_{\alpha}\nabla^{\alpha}\overset{2}{G}$ from the $\boldsymbol{F}_{\alpha}\nabla^{\alpha}\overset{2}{G}$ term in (29). After performing this elimination and using (9) to express $\overset{2}{\boldsymbol{E}}$ in terms of $\overset{1}{\boldsymbol{E}}$, we finally obtain

$$\begin{aligned}\frac{1}{2}\overset{2}{\boldsymbol{E}}(\widetilde{S})=&\fint_{\Sigma}\left[\overset{r}{\mu}\overset{2}{G}\frac{\partial\overset{1}{\boldsymbol{E}}}{\partial N}-\overset{1}{\boldsymbol{E}}\frac{\partial\overset{2}{G}}{\partial N}\right]\mathrm{d}S\\&+(\overset{r}{\mu}-\overset{r}{\epsilon}^{-1})\fint_{\Sigma}(\boldsymbol{N}\cdot\overset{1}{\boldsymbol{E}})\nabla\overset{2}{G}\,\mathrm{d}S\\&+(1-\overset{r}{\mu})\fint_{\Sigma}(\overset{1}{\boldsymbol{E}}\cdot\nabla\overset{2}{G})\boldsymbol{N}\,\mathrm{d}S\end{aligned} \tag{31a}$$

where

$$\overset{2}{\boldsymbol{E}}=\overset{1}{\boldsymbol{E}}+(\overset{r}{\epsilon}^{-1}-1)(\boldsymbol{N}\cdot\overset{1}{\boldsymbol{E}})\boldsymbol{N}. \tag{31b}$$

Note that (31b) immediately follows from (10) after it is contracted with $\boldsymbol{Z}_{i}$. Equation (31) together with (6a) form the desired set of integral equations. Ordinarily, this set can be solved for $\overset{1}{\boldsymbol{E}}$ and $\partial\overset{1}{\boldsymbol{E}}/\partial N$ via well-known numerical techniques such as the method of moments (Galerkin's method) [27, Ch. 4], [32], [33] or the locally corrected Nyström method [34].

To gain some insight into (31), let us consider a common scattering situation when Regions 1 and 2 have the same permeability, i.e., $\overset{r}{\mu}=1$. Under this condition, (6a) remains the same but (31a) becomes

$$\frac{1}{2}\overset{2}{\boldsymbol{E}}(\widetilde{S})=\fint_{\Sigma}\left[\overset{2}{G}\frac{\partial\overset{1}{\boldsymbol{E}}}{\partial N}-\overset{1}{\boldsymbol{E}}\frac{\partial\overset{2}{G}}{\partial N}\right]\mathrm{d}S+(1-\overset{r}{\epsilon}^{-1})\fint_{\Sigma}(\boldsymbol{N}\cdot\overset{1}{\boldsymbol{E}})\nabla\overset{2}{G}\,\mathrm{d}S. \tag{32}$$

If we discard the last integral on the RHS of (32), then (6a) and (32) are of the form that is traditionally used in scalar diffraction theory [27, Sec. 2.1], [35, Sec. 10.5]. In other words, it is the normal component of the E-field that separates the scalar and the vector surface integral equations.

To further illustrate this point, consider scattering of a z-polarized plane wave by a cylinder of arbitrary cross section whose axis is parallel to the z-axis. Under these conditions, the E-field only has a z-component, whence $\boldsymbol{N}\cdot\boldsymbol{E}=0$, and therefore (32) reduces to

$$\frac{1}{2}\overset{2}{\boldsymbol{E}}(\widetilde{S})=\fint_{\Sigma}\left[\overset{2}{G}\frac{\partial\overset{1}{\boldsymbol{E}}}{\partial N}-\overset{1}{\boldsymbol{E}}\frac{\partial\overset{2}{G}}{\partial N}\right]\mathrm{d}S \tag{33}$$

as expected. Of course, the integral in (33) is now interpreted as an integral over the perimeter of the cylinder and not as a surface integral.

In the sense discussed earlier, we can think of (6a) and (31) as a natural extension of the scalar diffraction theory. In other words, the RHS of (31a) contains three integrals that loosely can be associated with scalar diffraction, surface charge, and magnetic contributions, respectively.

V. Stratton–Chu Formula

In Section II, we derived the integral representation of the E-field (see (3)) directly from the vector Helmholtz equation. From this simple and streamlined derivation, the connection to the Stratton–Chu formula [36], [37, Sec. 8.14]

$$\boldsymbol{E}=-\int_{\Sigma}\Big[\mathrm{i}\frac{\mu\omega}{c}(\boldsymbol{N}\times\boldsymbol{H})\,G+(\boldsymbol{N}\cdot\boldsymbol{E})\,\nabla G+(\boldsymbol{N}\times\boldsymbol{E})\times\nabla G\Big]\mathrm{d}S \tag{34}$$

may not be obvious to the reader. To see the connection between (3) and (34), multiply (12) by G to obtain

$$-\mathrm{i}\frac{\mu\omega}{c}(\boldsymbol{N}\times\boldsymbol{H})\,G-(\boldsymbol{N}\cdot\boldsymbol{E})\,\nabla G=G\frac{\partial\boldsymbol{E}}{\partial N}-N_{m}\nabla\mathcal{A}^{m} \tag{35}$$

where $\mathcal{A}^m = E^m G$. We see that (35) takes care of the first two terms on the RHS of (34), and the last term on the RHS of (34) can be expanded via

$$(\boldsymbol{N} \times \boldsymbol{E}) \times \boldsymbol{\nabla} G = \left(\delta^i_m \delta^k_\ell - \delta^i_\ell \delta^k_m\right) N^\ell E^m \nabla_k G \boldsymbol{Z}_i = \boldsymbol{E}\frac{\partial G}{\partial N} - \boldsymbol{N}\nabla_m \mathcal{A}^m + \boldsymbol{N} G \nabla_m E^m. \quad (36)$$

Putting (35) and (36) into (34) yields

$$\boldsymbol{E} = \int_\Sigma \left[G\frac{\partial \boldsymbol{E}}{\partial N} - \boldsymbol{E}\frac{\partial G}{\partial N} - \boldsymbol{N}\left(\nabla \cdot \boldsymbol{E}\right) G \right] \mathrm{d}S + \boldsymbol{\mathcal{O}} \quad (37a)$$

where

$$\boldsymbol{\mathcal{O}} = \int_\Sigma [N_\ell (\boldsymbol{Z}^\ell \nabla_m \mathcal{A}^m) - N_m (\boldsymbol{Z}^\ell \nabla_\ell \mathcal{A}^m)] \mathrm{d}S. \quad (37b)$$

However, in Euclidean space (37b) vanishes. To see this, apply Gauss's theorem to (37b), then rewrite the result in terms of the Riemann–Christoffel tensor $R^k_{\cdot i\ell m}$, i.e., $(\nabla_\ell \nabla_m - \nabla_m \nabla_\ell)\,\mathcal{A}^k = R^k_{\cdot i\ell m}\mathcal{A}^i$, and note that the Riemann–Christoffel tensor vanishes in Euclidean space [28, Sec. 8.8], [29, Sec. 12.6]. In other words

$$\boldsymbol{\mathcal{O}} = \int_{\text{Vol.}} \boldsymbol{Z}^\ell [\nabla_\ell \nabla_m - \nabla_m \nabla_\ell] \mathcal{A}^m \,\mathrm{d}V = \int_{\text{Vol.}} \boldsymbol{Z}^\ell R^m_{\cdot i\ell m} \mathcal{A}^i \,\mathrm{d}V = \boldsymbol{0}. \quad (38)$$

Finally, we note that $\boldsymbol{\nabla} \cdot \boldsymbol{E}$ vanishes even on the boundary Σ (because there are no primary sources on Σ) and (37a) reduces to (3). Therefore, we conclude that (3) is equivalent to the Stratton–Chu formula with $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$ explicitly enforced on the boundary.

VI. Conclusion

We present an alternative set of surface integral equations for EM scattering, where the electric field and its normal derivatives are chosen as the boundary unknowns (see (6a) and (31)). To derive these, we develop a continuity condition for the normal derivative of the electric field (see (19)). We obtain this condition directly from the time-harmonic Maxwell equations and the conventional continuity conditions on the EM fields. Throughout this paper, we have relied on tensor calculus to keep the results independent of a particular coordinate system and not to obscure physical/geometrical interpretation. This also allows us to identify a number of intriguing relationships involving closed surface integrals of the electric field (see (24), (26), (27), and (28)).

In conclusion, we remark that analogous relationships exist for the H-field as well. For the reader's convenience, we summarize them in Appendix C. Note that these relationships may be obtained via $\boldsymbol{E} \Rightarrow \boldsymbol{H}$, $\boldsymbol{H} \Rightarrow -\boldsymbol{E}$, and $\epsilon \Leftrightarrow \mu$ replacement rules.

Appendix A
Tensor Calculus

A. Projection Decomposition Formula

To derive the projection decomposition formula used in the body of this paper, we note that any vector $\boldsymbol{A}$ can be written in terms of its normal and tangential components, that is,

$$\boldsymbol{A} = (\boldsymbol{N} \cdot \boldsymbol{A})\boldsymbol{N} + (\boldsymbol{S}^\alpha \cdot \boldsymbol{A})\boldsymbol{S}_\alpha. \quad (A.1)$$

Writing (A.1) in the component form

$$A^i \boldsymbol{Z}_i = (N_j A^j) N^k \boldsymbol{Z}_k + \left(Z^\alpha_i A^i\right) Z^k_\alpha \boldsymbol{Z}_k \quad (A.2)$$

and taking an inner product with $\boldsymbol{Z}^\ell$ yields

$$A^i \left[\delta^\ell_i - N_i N^\ell - Z^\alpha_i Z^\ell_\alpha\right] = 0 \quad (A.3)$$

where we used the fact that $\boldsymbol{Z}^\ell \cdot \boldsymbol{Z}_i = \delta^\ell_i$. Of course, (A.3) must hold for an arbitrary vector $\boldsymbol{A}$, and thus we have the desired projection decomposition formula

$$N^i N_j + Z^i_\alpha Z^\alpha_j = \delta^i_j. \quad (A.4)$$

B. Covariant Derivative

The covariant derivative ∇_i is defined by

$$\nabla_i A^j = \frac{\partial A^j}{\partial Z^i} + \Gamma^j_{ik} A^k \quad (A.5a)$$

and

$$\nabla_i A_j = \frac{\partial A_j}{\partial Z^i} - \Gamma^k_{ij} A_k \quad (A.5b)$$

where the Christoffel symbol Γ^k_{ij} is given by

$$\frac{\partial \boldsymbol{Z}_i}{\partial Z^j} = \Gamma^k_{ij} \boldsymbol{Z}_k. \quad (A.6)$$

Applying ∇_i to $\boldsymbol{Z}_j$ and using (A.5b) immediately yields $\nabla_i \boldsymbol{Z}_j = \boldsymbol{0}$. Similarly, one can show that $\nabla_i \boldsymbol{Z}^j = \boldsymbol{0}$. We follow Grinfeld [28, Sec. 8.6.7] and refer to these as the *metrinilic* property. Sometimes it is useful to have an explicit equation for the Christoffel symbol. This can be accomplished by taking an inner product of $\boldsymbol{Z}^\ell$ and (A.6) to obtain

$$\Gamma^\ell_{ij} = \boldsymbol{Z}^\ell \cdot \frac{\partial \boldsymbol{Z}_i}{\partial Z^j}. \quad (A.7)$$

C. Covariant Surface Derivative

The covariant surface derivative ∇_α is defined by

$$\nabla_\alpha A^\beta = \frac{\partial A^\beta}{\partial Z^\alpha} + \Gamma^\beta_{\alpha\gamma} A^\gamma \quad (A.8a)$$

and

$$\nabla_\alpha A_\beta = \frac{\partial A_\beta}{\partial Z^\alpha} - \Gamma^\gamma_{\alpha\beta} A_\gamma \quad (A.8b)$$

where the Christoffel symbol $\Gamma^\gamma_{\alpha\beta}$ on an embedded surface is given by

$$\Gamma^\gamma_{\alpha\beta} = \boldsymbol{S}^\gamma \cdot \frac{\partial \boldsymbol{S}_\alpha}{\partial S^\beta}. \quad (A.9)$$

Note that a definition analogous to (A.6) is not possible because $\partial \boldsymbol{S}_\alpha / \partial S^\beta$ may have components that do not lie in the tangent plane of the surface. In fact, if we apply (A.8b) to $\boldsymbol{S}_\beta$ and substitute the result into $\boldsymbol{S}^\gamma \cdot \nabla_\alpha \boldsymbol{S}_\beta$, we obtain $\boldsymbol{S}^\gamma \cdot \nabla_\alpha \boldsymbol{S}_\beta = 0$. In other words, we see that $\nabla_\alpha \boldsymbol{S}_\beta$ lies along the normal vector $\boldsymbol{N}$, that is,

$$\nabla_\alpha \boldsymbol{S}_\beta = W_{\alpha\beta} \boldsymbol{N} \quad (A.10)$$

where $W_{\alpha\beta}$ is termed the curvature tensor. The covariant surface derivative is metrinilic with respect to $\boldsymbol{Z}^i$ and $\boldsymbol{Z}_i$ but *not* with respect to $\boldsymbol{S}^\alpha$ and $\boldsymbol{S}_\alpha$. Thus, substituting $\boldsymbol{S}_\beta = Z^i_\beta \boldsymbol{Z}_i$ and $\boldsymbol{N} = N^i \boldsymbol{Z}_i$ into (A.10) immediately yields

$$\nabla_\alpha Z^i_\beta = N^i W_{\alpha\beta}. \tag{A.11}$$

D. Gauss's Theorem

Undoubtedly, the reader is familiar with Gauss's theorem in three dimensions, which relates a volume integral to a closed surface integral. However, the reader may be less familiar with Gauss's theorem in two dimensions, which relates a surface integral over a curved surface patch to an integral over the boundary of that patch. Before we state this theorem, we briefly review the elementary aspects of embedded curves. A curve can be viewed as an object embedded in a 3-D Euclidean space, or as a hypersurface embedded in a surface (non-Euclidean space). In the former view point, the codimension of the curve is $3 - 1 = 2$. Thus, the normal space is a plane and the curve does not have a unique normal direction at each point on a curve. In the latter view point, the codimension of the curve is $2 - 1 = 1$, and thus the curve has a well-defined normal at each point on the curve. This unit normal is given by

$$\boldsymbol{n} = n^\alpha \boldsymbol{S}_\alpha \tag{A.12}$$

where $\boldsymbol{S}_\alpha$ is the surface covariant basis. In other words, the normal $\boldsymbol{n}$ is orthogonal to the curve and lies in the *tangent* plane to the surface.

Gauss's theorem in two dimensions is given by [28, Sec. 14.5]

$$\int_\Sigma \nabla_\alpha A^\alpha \,\mathrm{d}S = \int_{\partial\Sigma} n_\alpha A^\alpha \,\mathrm{d}C \tag{A.13}$$

where Σ is a surface patch bounded by a contour $\partial\Sigma$. To illustrate the predominant use of (A.13), in this paper, we apply it to (A.10). Raising and contracting the index β with the α index in (A.10), and then applying (A.13) yields

$$\int_\Sigma W^\alpha_\alpha \boldsymbol{N}\,\mathrm{d}S = \int_\Sigma \nabla_\alpha \boldsymbol{S}^\alpha \,\mathrm{d}S = \int_{\partial\Sigma} n_\alpha \boldsymbol{S}^\alpha \,\mathrm{d}C = \int_{\partial\Sigma} \boldsymbol{n}\,\mathrm{d}C. \tag{A.14}$$

If we let Σ be a *closed* surface, then $\partial\Sigma$ does *not* exists (strictly speaking, it is a curve of measure zero) and the RHS of (A.14) vanishes, i.e., $\int_\Sigma W^\alpha_\alpha \boldsymbol{N}\,\mathrm{d}S = 0$ for any closed surface. Furthermore, it has been noted by Grinfeld [28, p. 244] that (A.14) "gives a vivid geometric interpretation of mean curvature: mean curvature measures the degree to which the contour boundary normal field $\boldsymbol{n}$ points consistently out of the plane."

Appendix B
Proof of $E^m \nabla^\alpha (N_\ell Z_{m\alpha} - N_m Z_{\ell\alpha}) = 0$

To prove that

$$E^m \nabla^\alpha (N_\ell Z_{m\alpha} - N_m Z_{\ell\alpha}) = 0 \tag{B.1}$$

we first use the product rule to expand (B.1) and then relate $\nabla^\alpha Z_{i\alpha}$ to the curvature tensor via (A.11), to obtain

$$E^m \nabla^\alpha (N_\ell Z_{m\alpha} - N_m Z_{\ell\alpha}) = E^m (Z_{m\alpha}\nabla^\alpha N_\ell - Z_{\ell\alpha}\nabla^\alpha N_m).$$

Using Weingarten's formula $\nabla_\alpha N^i = -Z^i_\beta W^\beta_\alpha$ [28, Sec. 11.9] on the RHS yields

$$E^m \nabla^\alpha (N_\ell Z_{m\alpha} - N_m Z_{\ell\alpha}) = E_m (Z^\ell_\alpha Z^m_\beta - Z^\ell_\beta Z^m_\alpha) W^{\beta\alpha}$$

where the RHS vanishes because the curvature tensor is symmetric, i.e., $W^{\alpha\beta} = W^{\beta\alpha}$ [28, Sec. 12.4].

Appendix C
H-Field Formulation

$$\overset{2}{H}{}^i = C^i_j \overset{1}{H}{}^j, \text{ where } C^i_j = \delta^i_j + (\overset{r}{\mu}{}^{-1} - 1) N^i N_j \tag{C.1}$$

$$\frac{\partial \boldsymbol{H}}{\partial N} = \nabla^\alpha [(\boldsymbol{N}\cdot\boldsymbol{H})\boldsymbol{S}_\alpha - (\boldsymbol{S}_\alpha\cdot\boldsymbol{H})\boldsymbol{N}] + \mathrm{i}\frac{\epsilon\omega}{c}\boldsymbol{N}\times\boldsymbol{E} \tag{C.2}$$

$$\int_\Sigma \boldsymbol{N}\cdot\frac{\partial \boldsymbol{H}}{\partial N}\,\mathrm{d}S = \int_\Sigma (\boldsymbol{N}\cdot\boldsymbol{H}) W^\alpha_\alpha \,\mathrm{d}S \tag{C.3}$$

$$\int_\Sigma \boldsymbol{S}^\beta \cdot \frac{\partial \boldsymbol{H}}{\partial N}\,\mathrm{d}S = \int_\Sigma (\boldsymbol{S}_\alpha\cdot\boldsymbol{H}) W^{\alpha\beta}\,\mathrm{d}S + \mathrm{i}\frac{\epsilon\omega}{c}\int_\Sigma \varepsilon^{\alpha\beta}(\boldsymbol{S}_\alpha\cdot\boldsymbol{E})\,\mathrm{d}S \tag{C.4}$$

$$\int_\Sigma \frac{\partial \boldsymbol{H}}{\partial N}\,\mathrm{d}S = \mathrm{i}\frac{\epsilon\omega}{c}\int_\Sigma \boldsymbol{N}\times\boldsymbol{E}\,\mathrm{d}S \tag{C.5}$$

$$\frac{1}{\overset{2}{\epsilon}}\int_\Sigma \frac{\partial \overset{2}{\boldsymbol{H}}}{\partial N}\,\mathrm{d}S = \frac{1}{\overset{1}{\epsilon}}\int_\Sigma \frac{\partial \overset{1}{\boldsymbol{H}}}{\partial N}\,\mathrm{d}S \tag{C.6}$$

$$\frac{1}{2}\overset{2}{\boldsymbol{H}}(\widetilde{S}) = \fint_\Sigma \left[\overset{r}{\epsilon}\overset{2}{G}\frac{\partial \overset{1}{\boldsymbol{H}}}{\partial N} - \overset{1}{\boldsymbol{H}}\frac{\partial \overset{2}{G}}{\partial N}\right]\mathrm{d}S + (\overset{r}{\epsilon} - \overset{r}{\mu}{}^{-1})\fint_\Sigma (\boldsymbol{N}\cdot\overset{1}{\boldsymbol{H}})\nabla\overset{2}{G}\,\mathrm{d}S + (1 - \overset{r}{\epsilon})\fint_\Sigma (\overset{1}{\boldsymbol{H}}\cdot\nabla\overset{2}{G})\boldsymbol{N}\,\mathrm{d}S \tag{C.7a}$$

where

$$\overset{2}{\boldsymbol{H}} = \overset{1}{\boldsymbol{H}} + (\overset{r}{\mu}{}^{-1} - 1)(\boldsymbol{N}\cdot\overset{1}{\boldsymbol{H}})\boldsymbol{N}. \tag{C.7b}$$

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