Final Project

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1 What is inverse scattering?

In the mid-1960s, an important new class of transformation techniques was discovered. This transform, called **scattering transform**, is capable of producing exact solutions for a small group of nonlinear evolution equations.

The idea of this transform is to view the solution $q(x,t)$ of the equation to be solved as the potential of the Schrödinger equation

$$-\frac{d^2u}{dx^2} + q(x,t)u = \lambda u.$$

For each fixed time t, we study the scattering problem and find the reflection and transmission coefficients and the bound states. Then, as time evolves, the potential $q(x,t)$ (and hence its scattering data) evolves as well. Now, the amazing thing we discover is that there are certain equations for which the evolution of the scattering data is diagonalized, that is, the scattering data for a given frequency evolves independent of all other frequencies. Thus, instead of studying directly the evolution of $q(x,t)$, we study the evolution of the scattering data, which is described by separated, linear differential equations whose solutions are easily obtained, and from this information we can construct $q(x,t)$.

To make any transformation work, we must know how to transform functions and also how to invert the transform. For the Schrödinger equation, we know how, at least in principle, to transform potentials $q(x)$ by finding scattering data. A natural question to ask is if one can reconstruct a potential $q(x)$ from knowledge of its scattering data. This is an important problem, not only in the context of the Schrödinger equation, but in many other applications such as x-ray tomography, seismic exploration, radar imaging, medical imaging, etc., where the shape of an object must be determined from indirect information, such as the way waves are reflected or transmitted by the object. We will discuss only the problem of reconstructing the potential of the Schrödinger equation, with the realization that this is but a small part of the important field of inverse imaging problems in which there is still much active research and development.[1]

2 You want me to do what?

The final project will consist of you finding the width and the height of the rectangular barrier potential by doing a curve-of-best-fit to the transmission coefficient using *Mathematica*™. After you find the curve, you will write a **journal quality** LaTeX document describing what you have done; hence why Section 1 was included.

2.1 How about you give me some theory and equations to use?

The time-independent Schrödinger equation in one dimension is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi, \tag{1}$$

where ψ is the wave function, E is the energy, and V is the potential. It's common practice to work with dimensionless quantities by scaling (1). Formally this can be done with the Buckingham Π theorem, however, we will use the following ad hoc scaling

$$V_s = \frac{V}{\frac{\hbar^2}{2mb^2}}, \qquad E_s = \frac{E}{\frac{\hbar^2}{2mb^2}}, \qquad x_s = \frac{x}{b}, \tag{2}$$

where $b = 2\ell$. Notice that ℓ in (2) sets the scale for the measure of length and that E_s, V_s are dimensionless quantities if ℓ is measured in units of meter because $\frac{\hbar^2}{2mb^2}$ is measured in units of $\mathrm{J^2{\cdot}s^2}/\mathrm{kg{\cdot}m^2} = \mathrm{J}$. Schrödinger equation (1) becomes

$$-\frac{d^2\psi}{dx_s^2} + V_s\psi = E_s\psi \tag{3}$$

in our scaled units, where ψ and V_s are now functions of x_s. The rectangular barrier potential is given by

$$V_s(x_s) = \begin{cases} 0 & : \quad x_s \leq -\ell \\ V_o & : \quad -\ell < x_s < \ell \\ 0 & : \quad x_s \geq \ell \end{cases} \tag{4}$$

[1]The above was taken almost verbatim from [1].

where V_o is the height and 2ℓ is the width of the rectangular barrier potential, see Fig. 1. The transmission coefficient for the rectangular barrier potential is given by [2]

$$T = \frac{1}{1 + \frac{V_o^2}{4E_s(E_s - V_o)} \sin^2 \sqrt{E_s - V_o}}. \tag{5}$$

Notice that ℓ does not appear in (5). Thus, we will not be able to determine the width of the rectangular barrier potential from the transmission coefficient.

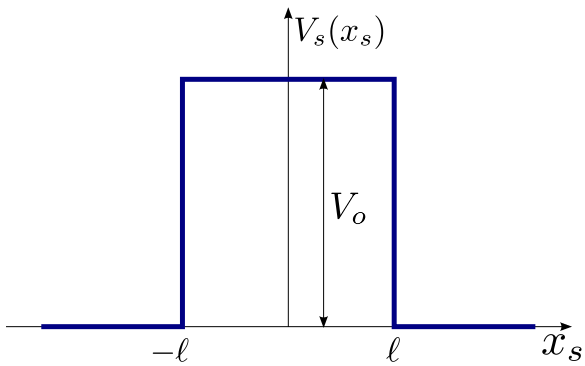

Figure 1: The rectangular barrier potential of width 2ℓ and height V_o is shown.

The formalism above can be extended to *multiple* rectangular barrier potentials by the **transfer matrix**,

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{23} \\ M_{21} & M_{22} \end{pmatrix}. \tag{6}$$

For example, if a potential consists of two non-overlapping rectangular barrier potentials then the $\mathbf{M}$-matrix for the combination is the product of the two $\mathbf{M}$-matrices,

$$\mathbf{M} = \mathbf{M}_2 \mathbf{M}_1,$$

with an obvious generalization for N potentials,

$$\mathbf{M} = \mathbf{M}_N \mathbf{M}_{N-1} \cdots \mathbf{M}_2 \mathbf{M}_1. \tag{7}$$

2.2 So, what I'm supposed to fit to what?

Fit equation (5) to `dataSet2.dat`, which is available from Yuffa's Page. You will find V_o in (5) by doing `FindFit` in *Mathematica*™, a reasonable range for V_o is $5 < V_o < 15$. After you find the curve-of-best-fit you will need to plot it. A nice looking plot range is `PlotRange -> {{20, 100}, {.95, 1.01}}` for `dataSet2.dat`. Hint: In (5), E_s plays the role of "x" and T plays the role of "y". Also, note that the parameter you are trying to find is V_o subject to constraint, $5 < V_o < 15$.

3 What are the requirements section by section?

You will use the field session preamble for your LaTeX document, see p.7 of LaTeX 101. Don't forget to include a table of contents and a list of figures, as well as an abstract. You can typeset an abstract in LaTeX by using `\abstract{Your abstract goes here}`, right after a list of figures. Obviously, the abstract should include a brief summary of your final project; few sentences should do.

3.1 Introduction

Your report should have an "Introduction" section. For the "Introduction" section you can simply use Section 1 of this document. Feel free to re-phrase it if you want.

3.2 Theory

Your report should have a "Theory" section. For the "Theory" section you can simply use Section 2.1 of this document. You can change the wording if you wish but all equations and units as well as the figure must be present. Don't even think about ripping the figure out of this PDF; you must re-draw it in Inkscape and save it in `.pdf` format (don't forget to do `fit-to-page`).

3.3 Experiment

Your report should have an "Experiment" section. Write any story you want about how the `dataSet2.dat` was obtained. You should also include a photograph of the experimental set-up in your "Experiment" section. Use Google to find some experimental set-up that looks cool. You may find yourself needing to convert between different bitmap formats, to do this type `convert filename.oldFormat filename.newFormat` into the Linux terminal. Recall that `pdflatex` only supports `.png` and `.jpg` bitmap formats.

3.4 Data Analysis

Your report should have a "Data Analysis" section. Write a reasonable story about how well your curve-of-best-fit matches the data in `dataSet2.dat`. Make sure to include a plot that contains your curve-of-best-fit and data with both x and y error bars. The plot should also have some "pretty" LaTeX formula (such as equation (5)), plot title, and etc.

3.5 Conclusion

Your report should have a "Conclusion" section. Write one to two paragraphs summarizing your final project. Don't forget to state that the width of the rectangular barrier potential could not be determined from the transmission coefficient.

3.6 About the Author

Your report should have "About the Author" section. Write a few words about yourself and include a picture of your beautiful self (if you have it).

3.7 References

Your report should have a "References" section. For the "References" you can use the references at the end of this document or make-up your own. Make sure to reference, `\cite{yourLabel}`, at least two different books somewhere in your document.

3.8 FAQ

- `Schr{\"o}dinger` Schrödinger
- `$\ell$` ℓ
- `$\unitfrac{J^2 \cdot s^2}{kg \cdot m^2}=\unit{J}$` $^{\mathrm{J^2 \cdot s^2}}/_{\mathrm{kg \cdot m^2}} = \mathrm{J}$
- Spacing: `$a \quad b$` $a \quad b$ or `$a \qquad b$` $a \qquad b$
- `\textit{Mathematica}\texttrademark` *Mathematica*™
- Don't worry about making any of the numbers or section titles blue and hyperlinked.
- You should print your final PDF and turn it in for grading. Also, you should tar-up all files used in the production of your final project and copy the zipped tar file into my directory, i.e.,

```
tar -czvf usernameFinal.tgz yourFinalProjectDirectory
cp usernameFinal.tgz  /Net/voodoo/home/ayuffa/FieldSession/
```

References

[1] James P. Keener, *Principles of Applied Mathematics,* (Perseus Books, 2000).

[2] David J. Griffiths, *Introduction to Quantum Mechanics,* (Prentice Hall, 1995).