1. An electromagnetic wave of wavelength λ is traveling in vacuum in the **negative** $z-$direction. The electric field has amplitude E_0 and is parallel to the $x-$axis.

(a) Find the electric field, $\vec{E}$, as a function of position and time in terms of the given parameters.

Solution: The electric field has amplitude E_0 and is parallel to the $x-$axis, thus, $\vec{E}$ is given by

$$\vec{E}(z,t) = E_0 \cos(kz + \omega t)\,\hat{\boldsymbol{\imath}}, \tag{1}$$

[Why did we use $+\omega t$ in (1)?] where k and ω in (1) can be found from their definitions, i.e.,

$$k = \frac{2\pi}{\lambda} \tag{2}$$

$$\omega = 2\pi f, \tag{3}$$

where $f = \frac{c}{\lambda}$. Substituting (2) and (3) into (1) yields

$$\vec{E}(z,t) = E_0 \cos\left(\frac{2\pi}{\lambda}z + \frac{2\pi c}{\lambda}t\right)\hat{\boldsymbol{\imath}}.$$

(b) Find the magnetic field, $\vec{B}$, as a function of position and time in terms of the given parameters.

Solution: The wave is propagating in the **negative** $z-$direction and the electric field is parallel to the $x-$axis, so $\vec{B}$ must be in the **negative** $y-$direction because

$$\vec{S} = \frac{1}{\mu_o}\vec{E} \times \vec{B}.$$

Thus,

$$\begin{aligned}
\vec{B}(z,t) &= B_0 \cos(kz + \omega t)\,(-\hat{\boldsymbol{\jmath}}) \\
&= \frac{E_0}{c}\cos(kz + \omega t)\,(-\hat{\boldsymbol{\jmath}}) \\
\vec{B}(z,t) &= -\frac{E_0}{c}\cos\left(\frac{2\pi}{\lambda}z + \frac{2\pi c}{\lambda}t\right)\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

2. Electromagnetic radiation is emitted by accelerating charges. The rate at which energy is emitted from an accelerating charge that has charge q and acceleration a is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{q^2 a^2}{6\pi\epsilon_o c^3}, \tag{4}$$

where c is the speed of light.

(a) The electron in a hydrogen atom can be considered to be in circular orbit with a radius of $R = 0.0529$ nm and a kinetic energy of KE = 13.5 eV. If the electron behaved classically, how much energy would it radiate per second?

Solution: To express (4) in terms of the given parameters, we note that centripetal acceleration can be written in terms of kinetic energy, i.e.,

$$\begin{aligned} a &= \frac{v^2}{R} \\ &= \frac{2\mathrm{KE}}{mR}. \end{aligned} \tag{5}$$

Substituting (5) into (4) yields

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{q^2}{6\pi\epsilon_o c^3}\left(\frac{2\mathrm{KE}}{mR}\right)^2.$$

Thus,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 2.89 \cdot 10^{11}\ \mathrm{eV/s}. \tag{6}$$

(b) Can we use classical physics to describe the motion of an electron in a hydrogen atom?

Solution: From (6), we see that the electron will collapse onto the nucleus almost immediately. Thus, we must use a more elaborate theory to describe the motion of the electron, e.g., quantum mechanics. Surprisingly enough, this is a workhorse problem in quantum mechanics.

3. Light is incident along the normal on face AB of a glass prism of refractive index n_2, as shown in Fig. 1. Find the largest value the angle α can have without any light refracted out of the prism at face AC if the prism is immersed in water of refractive index n_1.

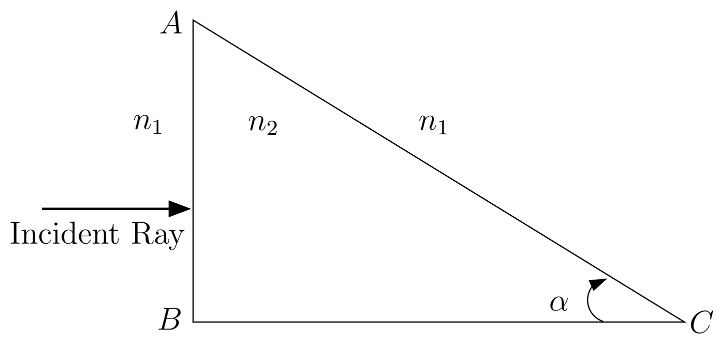

Figure 1: A glass prism is shown.

Solution: Since the light ray is incident along the normal on face AB, it is not bent at the AB interface (prove this). Applying Snell's law at the AC interface (see Fig. 2) yields

$$n_2 \sin\beta = n_1 \sin 90^\circ$$
$$n_2 \sin(90^\circ - \alpha) = n_1$$
$$\alpha = \arccos\left(\frac{n_1}{n_2}\right).$$

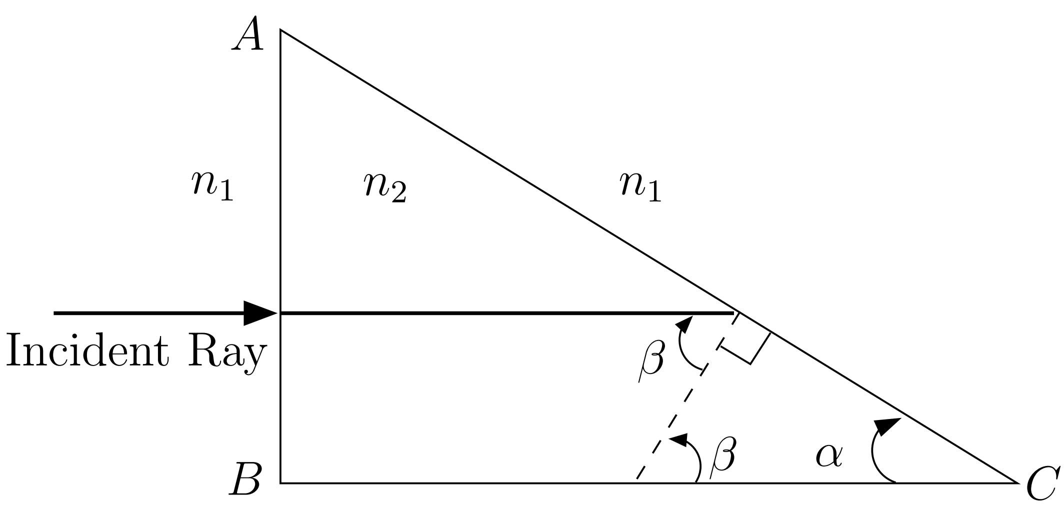

Figure 2: A glass prism is shown. From the right triangle, we obtain $\beta = 90^\circ - \alpha$.

4. Fermat's principle of least time states that "the path taken between two points by a ray of light is the path that can be traversed in the least time." More generally, such a principle is known as the "least action principle." This principle plays a fundamental role in modern physics, as well as in mathematics. Most of modern physics has been cast in terms of the principle of least action. No physics course is complete without at least a mention of the least action principle, hence, this problem.

   (a) A ray of light traveling with speed c leaves point 1 (assumed to be fixed), shown in Fig. 3, and is reflected to point 2 (assumed to be fixed). The ray strikes the reflecting surface a horizontal distance x from point 1. Using Fermat's principle of least time, show that $\theta_1 = \theta_2$.

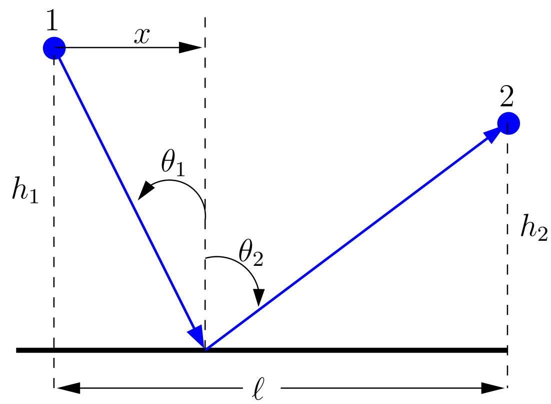

Figure 3: A ray of light is shown in blue.

Solution: Applying the Pythagorean theorem twice, we obtain the total distance traveled by the light ray (see Fig. 3),

$$d = \sqrt{{h_1}^2 + x^2} + \sqrt{{h_2}^2 + (\ell - x)^2}.$$

The time required for the light to travel from point 1 to point 2 is given by

$$t = \frac{d}{c}$$
$$t = \frac{\sqrt{{h_1}^2 + x^2} + \sqrt{{h_2}^2 + (\ell - x)^2}}{c}.$$

Applying Fermat's principle of least time yields

$$\frac{\mathrm{d}t}{\mathrm{d}x} = 0$$
$$\frac{x}{\sqrt{{h_1}^2 + x^2}} = \frac{\ell - x}{\sqrt{{h_2}^2 + (\ell - x)^2}}$$
$$\sin\theta_1 = \sin\theta_2$$
$$\theta_1 = \theta_2.$$

(b) A ray of light goes from point 1 (assumed to be fixed), in a medium in which the speed of light is v_1, to point 2 (assumed to be fixed), in a medium in which the speed of light is v_2, see Fig. 4. The ray strikes the interface a horizontal distance x to the right of point 1. Using Fermat's principle of least time, show that $n_1 \sin\theta_1 = n_2 \sin\theta_2$, i.e., Snell's law.

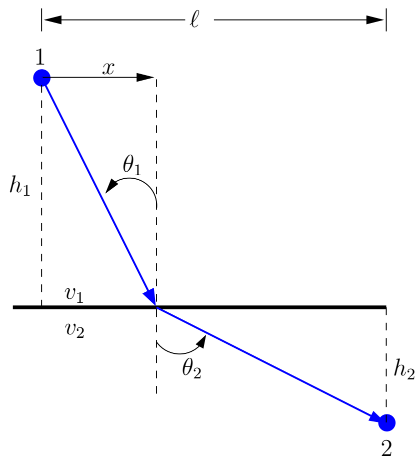

Figure 4: A ray of light is shown in blue.

Solution: Applying the Pythagorean theorem, we obtain the distance traveled by the light ray at speed v_1 (see Fig. 4)

$$d_1 = \sqrt{{h_1}^2 + x^2}.$$

Applying the Pythagorean theorem, we obtain the distance traveled by the light ray at speed v_2 (see Fig. 4)

$$d_2 = \sqrt{{h_2}^2 + (\ell - x)^2}.$$

The time required for the light to travel from point 1 to point 2 is given by

$$t = \frac{d_1}{v_1} + \frac{d_2}{v_2}$$

$$t = \frac{\sqrt{{h_1}^2 + x^2}}{v_1} + \frac{\sqrt{{h_2}^2 + (\ell - x)^2}}{v_2}$$

Applying Fermat's principle of least time yields

$$\frac{\mathrm{d}t}{\mathrm{d}x} = 0$$

$$\frac{x}{v_1\sqrt{{h_1}^2 + x^2}} = \frac{\ell - x}{v_2\sqrt{{h_2}^2 + (\ell - x)^2}}$$

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where we have used the fact that $nv = c$.