1. The cross-sections of two hollow infinitely long cylindrical conductors are shown in blue and red in Fig. 1. The blue cylindrical conductor carries a constant current, I_{12}, out of the page, with uniform current density J_{12}. The red cylindrical conductor carries a current into the page, with current density $J_{34} = \frac{3J_{12}r}{2}$.

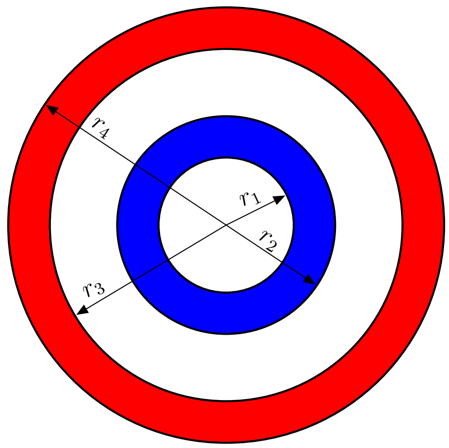

Figure 1: The cross-sections of two hollow infinitely long cylindrical conductors are shown in blue and red. The blue cylinder has inner radius r_1 and outer radius r_2. The red cylinder has inner radius r_3 and outer radius r_4.

(a) Find the magnitude of the magnetic field at a distance r, $r_1 < r < r_2$, from the axis.

Solution: We apply Ampere's Law, given by

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_o I_{\text{encl.}}, \tag{1}$$

at a distance r from the axis, see Fig. 2.

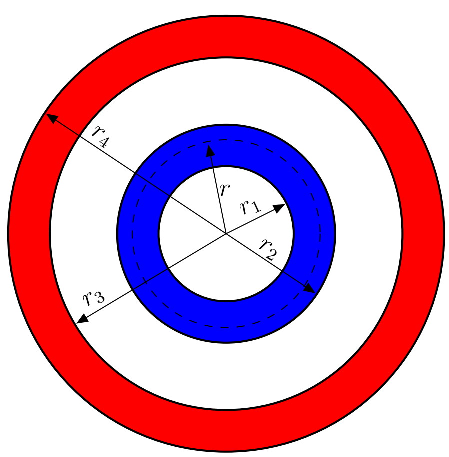

Figure 2: An Amperian loop of radius r is shown by the dashed circle.

By symmetry, $\vec{B}$ is parallel to $\mathrm{d}\vec{\ell}$; thus, (1) yields

$$\oint B\mathrm{d}\ell = \mu_o I_{\text{encl.}}. \tag{2}$$

By symmetry, the magnetic field B is constant on the Amperian loop; thus, (2) yields

$$\begin{aligned} B\oint \mathrm{d}\ell &= \mu_o I_{\text{encl.}} \\ B2\pi r &= \mu_o I_{\text{encl.}} \\ B &= \frac{\mu_o I_{\text{encl.}}}{2\pi r}, \end{aligned} \tag{3}$$

where $I_{\text{encl.}}$ refers to the current enclosed by the Amperian loop. $I_{\text{encl.}}$ can be found by introducing the current density and then integrating, i.e.,

$$\begin{aligned} I_{\text{encl.}} &= \int \vec{J}\cdot \mathrm{d}\vec{A} \\ &= \int J\,\mathrm{d}A, \qquad \text{(What happened to the dot product?)} \\ &= \int_{r_1}^{r} J_{12}\,(2\pi\zeta)\,\mathrm{d}\zeta \\ &= J_{12}\int_{r_1}^{r} 2\pi\zeta\,\mathrm{d}\zeta \\ &= J_{12}\pi\left(r^2 - {r_1}^2\right), \end{aligned} \tag{4}$$

where $J_{12} = \frac{I_{12}}{\pi({r_2}^2 - {r_1}^2)}$. Substituting (4) into (3) yields

$$B = \frac{\mu_o I_{12}}{2\pi r}\left(\frac{r^2 - {r_1}^2}{{r_2}^2 - {r_1}^2}\right).$$

(b) Find the magnitude of the magnetic field at a distance r, $r_3 < r < r_4$, from the axis.

Solution: We apply Ampere's Law, given by

$$\oint \vec{B}\cdot \mathrm{d}\vec{\ell} = \mu_o I_{\text{encl.}}, \tag{5}$$

at a distance r from the axis, see Fig. 3.

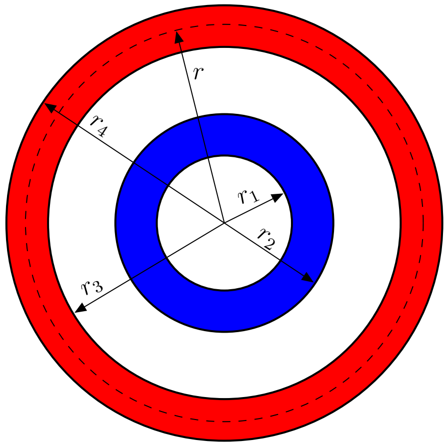

Figure 3: An Amperian loop of radius r is shown by the dashed circle.

By symmetry, $\vec{B}$ is parallel or anti-parallel to $\mathrm{d}\vec{\ell}$, i.e., $\vec{B}\cdot\mathrm{d}\vec{\ell} = +B\mathrm{d}\ell$ or $\vec{B}\cdot\mathrm{d}\vec{\ell} = -B\mathrm{d}\ell$; thus, we take the absolute value of both sides of (5) to yield

$$\left|\oint B\mathrm{d}\ell\right| = \left|\mu_o I_{\text{encl.}}\right|. \tag{6}$$

By symmetry, the magnetic field B is constant on the Amperian loop; thus, (6) yields

$$\begin{aligned}\left|B\oint \mathrm{d}\ell\right| &= \left|\mu_o I_{\text{encl.}}\right| \\ B &= \left|\frac{\mu_o I_{\text{encl.}}}{2\pi r}\right|, \end{aligned} \tag{7}$$

where $I_{\text{encl.}}$ refers to the current enclosed by the Amperian loop. $I_{\text{encl.}}$ can be found via current densities, i.e.,

$$\begin{aligned}\left|I_{\text{encl.}}\right| &= \left|\int \vec{J}\cdot\mathrm{d}\vec{A}\right| \\ &= \left|\int \vec{J}_{12}\cdot\mathrm{d}\vec{A}_{12} + \int \vec{J}_{23}\cdot\mathrm{d}\vec{A}_{23} + \int \vec{J}_{34}\cdot\mathrm{d}\vec{A}_{34}\right| \\ &= \left|\int J_{12}\,\mathrm{d}A_{12} + \int 0\,\mathrm{d}A_{23} - \int J_{34}\,\mathrm{d}A_{34}\right| \\ &= \left|\int_{r_1}^{r_2} J_{12}\left(2\pi\zeta\right)\mathrm{d}\zeta - \int_{r_3}^{r}\left(\frac{3J_{12}\zeta}{2}\right)\left(2\pi\zeta\right)\,\mathrm{d}\zeta\right| \\ &= I_{12}\left|1 - \frac{r^3 - r_3^3}{{r_2}^2 - {r_1}^2}\right|, \end{aligned} \tag{8}$$

where we have used the fact that $J_{12} = \frac{I_{12}}{\pi({r_2}^2 - {r_1}^2)}$. Substituting (8) into (7) yields

$$B = \frac{\mu_o I_{12}}{2\pi r}\left|1 - \frac{r^3 - {r_3}^3}{{r_2}^2 - {r_1}^2}\right|.$$

2. A blue wire is wound evenly on a torus of rectangular cross section, see Fig. 4. Find the self inductance of this torus.

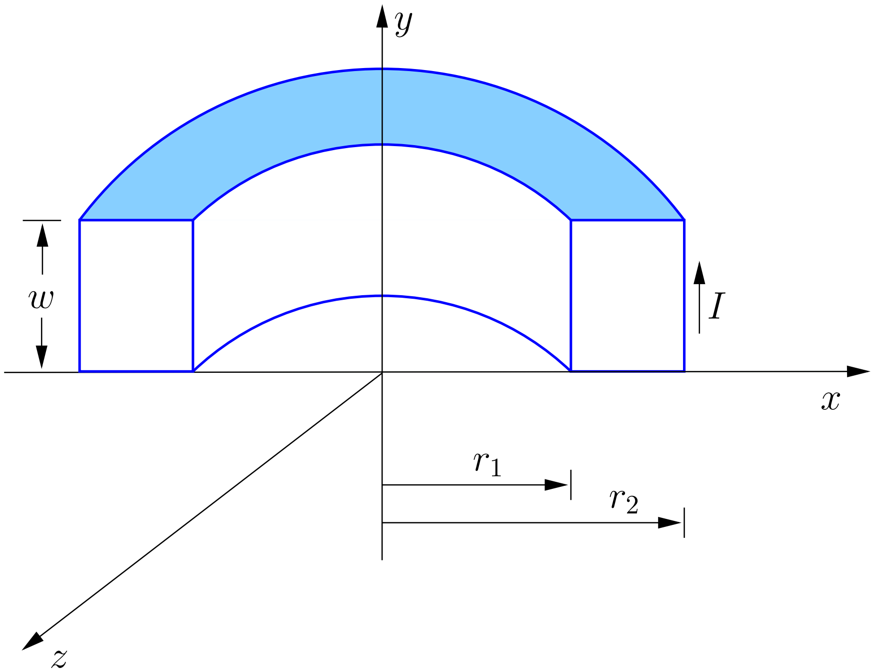

Figure 4: A blue wire is wound evenly on a torus of rectangular cross section, with inner radius r_1 and outer radius r_2. There are N turns of the blue wire in all.

Solution: The magnetic flux through one square loop is given by (see Recitation 6, problem 2)

$$\Phi_{\text{one loop}} = \frac{\mu_o N I w}{2\pi} \ln\left(\frac{r_2}{r_1}\right).$$

The magnetic flux through N square loops is given by

$$\Phi_{N \text{ loops}} = \frac{\mu_o N^2 I w}{2\pi} \ln\left(\frac{r_2}{r_1}\right),$$

thus, the self inductance of the torus is given by

$$L = \frac{\mu_o N^2 w}{2\pi} \ln\left(\frac{r_2}{r_1}\right).$$

3. A blue wire carrying current I is wound evenly on a torus of rectangular cross section. There are N turns of the blue wire in all. A red wire is thrown over the torus and is connected to a resistor, R, see Fig. 5. Find the mutual inductance of this arrangement.

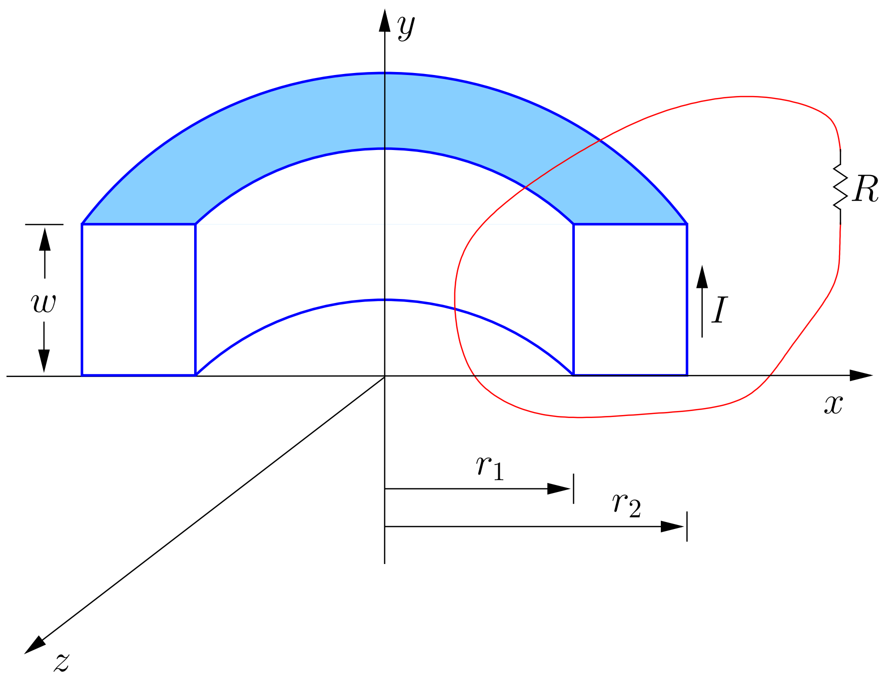

Figure 5: A blue wire carrying current I is wound evenly on a torus of rectangular cross section, with inner radius r_1 and outer radius r_2. There are N turns of the blue wire in all. A red wire is thrown over the torus and is connected to a resistor, R.

Solution: The magnetic flux through the **area enclosed by the red wire** is given by (see Recitation 6, problem 2)

$$\Phi_B = \frac{\mu_o N I w}{2\pi} \ln\left(\frac{r_2}{r_1}\right).$$

Thus, the mutual inductance is given by

$$M = \frac{\mu_o N w}{2\pi} \ln\left(\frac{r_2}{r_1}\right). \tag{9}$$

4. Consider the circuit shown in Fig. 6.

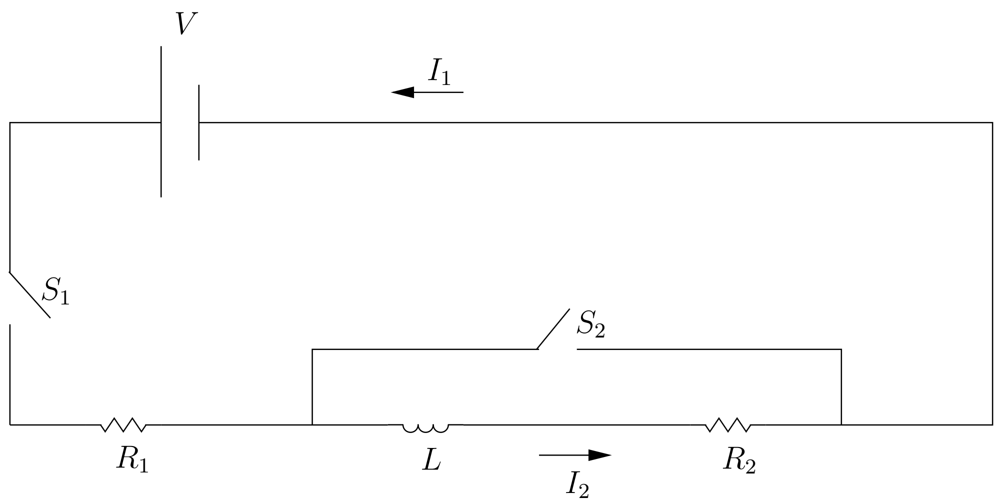

Figure 6: An LR circuit is shown.

(a) At time $t = 0$, switch S_1 is closed and switch S_2 is left open. Find current I_1 as a function of time.

Solution: Applying Kirchhoff's loop rule going in the counterclockwise direction through V, S_1, R_1, L, and R_2 yields

$$V - I_1 R_1 - L\frac{\mathrm{d}I_1}{\mathrm{d}t} - I_1 R_2 = 0$$

$$-\int \mathrm{d}t = \int \frac{L}{(R_1 + R_2)\, I_1 - V}\, \mathrm{d}I_1$$

$$-t + C_1 = \frac{L}{R_1 + R_2} \ln\left[(R_1 + R_2)\, I_1 - V\right], \quad \text{where } C_1 = \text{constant of integration}$$

$$I_1 = C_2 e^{-\frac{R_1+R_2}{L}t} + \frac{V}{R_1 + R_2}, \quad \text{where } C_2 = \text{constant.}$$

Using the initial conditions, i.e., $I_1 = 0$ at $t = 0$, yields

$$I_1 = \frac{V}{R_1 + R_2}\left(1 - e^{-\frac{R_1+R_2}{L}t}\right). \tag{10}$$

(b) Find current I_1 after a very long time.

Solution: Taking the limit as $t \to \infty$ of (10) yields

$$I_1 = \frac{V}{R_1 + R_2} \quad \text{at } t = \infty. \tag{11}$$

(c) After the current in the circuit has reached its final, steady-state value with switch S_1 closed and S_2 open, switch S_2 is closed, thus short-circuiting the inductor. (Switch S_1 remains closed.) Find current I_1.

Solution: Applying Kirchhoff's loop rule going in the counterclockwise direction through V, S_1, and S_2 yields

$$V - I_1 R_1 = 0$$
$$I_1 = \frac{V}{R_1}.$$

(d) Find current I_2 as a function of time t that has elapsed since S_2 was closed.

Solution: Applying Kirchhoff's loop rule going in the counterclockwise direction through L, R_2, and S_2 yields

$$-I_2 R_2 - L\frac{\mathrm{d}I_2}{\mathrm{d}t} = 0$$
$$-\int \frac{R_2}{L}\,\mathrm{d}t = \int \frac{1}{I_2}\,\mathrm{d}I_2$$
$$I_2 = Ce^{-\frac{R_2}{L}t}, \tag{12}$$

where C is some constant. At time $t = 0$, I_2 is given by (11) (why?), thus, (12) yields

$$I_2 = \frac{V}{R_1 + R_2}e^{-\frac{R_2}{L}t}.$$