1. Those damn infinite and semi-infinite (half-infinite) wires!

   (a) Consider a semi-infinite wire carrying current I, see Fig. 1. Find the magnetic field, $\vec{B}$, in the first quadrant, i.e., $x \geq 0, y > 0, z = 0$. You may find

$$\int_0^\infty \frac{\eta}{\left[(\zeta - w)^2 + \eta^2\right]^{3/2}} \, \mathrm{d}w = \frac{1}{\eta} + \frac{\zeta}{\eta\sqrt{\zeta^2 + \eta^2}} \tag{1}$$

useful.

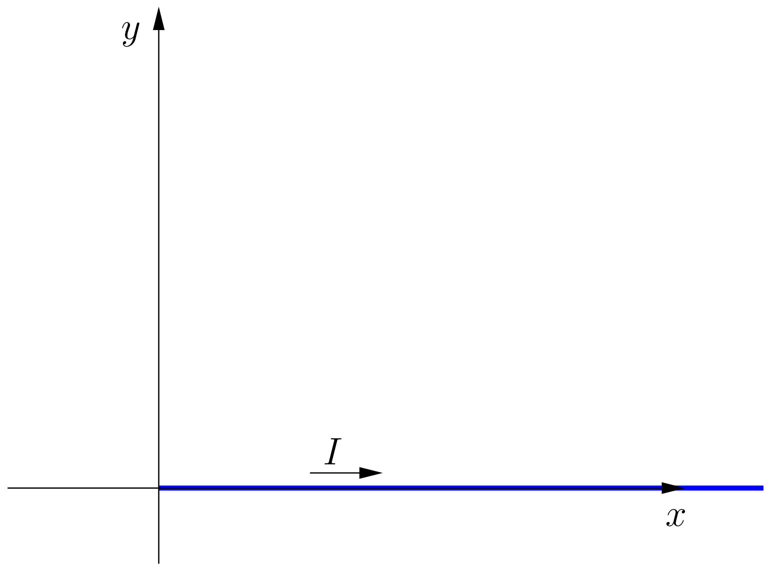

Figure 1: A semi-infinite wire (from 0 to ∞) carrying current I is shown in blue.

Solution: The magnetic field is given by

$$\vec{B} = \frac{\mu_o}{4\pi} \int \frac{I \, \mathrm{d}\vec{\ell} \times \vec{r}}{r^3}, \tag{2}$$

where $\vec{r} = \vec{r}_f - \vec{r}_s$, and $\mathrm{d}\vec{\ell}$ should really be $\mathrm{d}\vec{r}_s$. Against all of my wishes, we will NOT use $\mathrm{d}\vec{r}_s$ because your book and lecture notes call it $\mathrm{d}\vec{\ell}$ and it's important to have uniform notation. Recall that $\vec{r}_s$ is the **source** variable and $\vec{r}_f$ is the **field** variable. Computing $\vec{r}_f$ yields

$$\vec{r}_f = x \, \hat{\boldsymbol{\imath}} + y \, \hat{\boldsymbol{\jmath}}, \tag{3}$$

since we want to know the magnetic field anywhere in the first quadrant, i.e., $x \geq 0, y > 0, z = 0$. Computing $\vec{r}_s$ yields

$$\vec{r}_s = x_s \, \hat{\boldsymbol{\imath}}, \tag{4}$$

since the wire that produces the field lies on the x-axis [why is there an "s" subscript on x in (4)?]. Taking the difference between (3) and (4) yields

$$\vec{r} = (x - x_s) \, \hat{\boldsymbol{\imath}} + y \, \hat{\boldsymbol{\jmath}}. \tag{5}$$

$d\vec{\ell}$ is computed in the "usual way", i.e.,

$$\begin{aligned} \vec{\ell} &= x_s\,\hat{\boldsymbol{\imath}} \qquad \text{[compare this to (4)]} \\ \frac{d\vec{\ell}}{dx_s} &= \hat{\boldsymbol{\imath}} \\ d\vec{\ell} &= dx_s\,\hat{\boldsymbol{\imath}}. \end{aligned} \tag{6}$$

Substituting (5) and (6) into (2) yields

$$\vec{B} = \frac{\mu_o I}{4\pi}\int_0^\infty \frac{y}{\left[(x-x_s)^2+y^2\right]^{3/2}}\,dx_s\,\hat{\boldsymbol{k}}. \tag{7}$$

Finally, evaluating the integral in (7) via (1) yields

$$\vec{B} = \frac{\mu_o I}{4\pi}\left(\frac{1}{y}+\frac{x}{y\sqrt{x^2+y^2}}\right)\hat{\boldsymbol{k}}. \tag{8}$$

(b) Consider an infinite wire carrying current I, see Fig. 2. Find the magnetic field, $\vec{B}$, in the first quadrant, i.e., $x \geq 0, y > 0, z = 0$. You may find

$$\int_{-\infty}^{\infty} \frac{\eta}{\left[(\zeta-w)^2+\eta^2\right]^{3/2}}\,dw = \frac{2}{\eta} \tag{9}$$

useful.

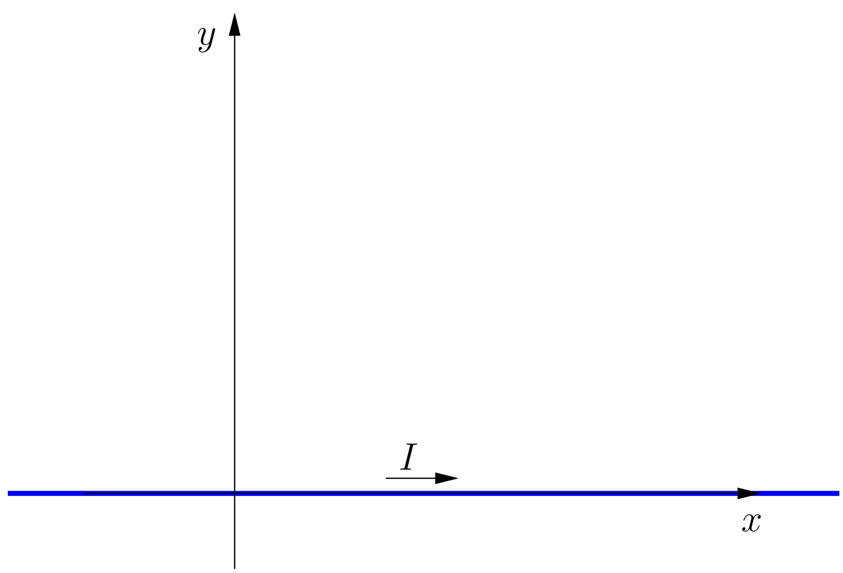

Figure 2: An infinite wire (from $-\infty$ to ∞) carrying current I is shown in blue.

Solution: The magnetic field due to an infinite wire is given by (7) but with different limits of integration [why?]. Thus,

$$\vec{B} = \frac{\mu_o I}{4\pi} \int_{-\infty}^{\infty} \frac{y}{\left[(x - x_s)^2 + y^2\right]^{3/2}} \, \mathrm{d}x_s \, \hat{\boldsymbol{k}}. \tag{10}$$

Finally, evaluating the integral in (10) via (9) yields

$$\vec{B} = \frac{\mu_o I}{2\pi y} \, \hat{\boldsymbol{k}}. \tag{11}$$

(c) Where does the magnetic field due to a semi-infinite wire equal one half the magnetic field of an infinite wire, i.e., $\vec{B}_{\text{semi-infinite}} = \vec{B}_{\text{infinite}}/2$?

Solution: The magnetic field due to a semi-infinite wire is equal to the magnetic field of an infinite wire divided by two when

$$\begin{aligned} \vec{B}_{\text{semi-infinite}} &= \frac{\vec{B}_{\text{infinite}}}{2} \\ \frac{\mu_o I}{4\pi} \left(\frac{1}{y} + \frac{x}{y\sqrt{x^2 + y^2}} \right) \hat{\boldsymbol{k}} &= \frac{\mu_o I}{4\pi y} \, \hat{\boldsymbol{k}} \\ \frac{x}{y\sqrt{x^2 + y^2}} &= 0 \\ x &= 0, \text{ i.e, on the } y\text{-axis.} \end{aligned}$$

What is so special about the y-axis?

2. A blue wire carrying current $I = I_o t^3/3$ is wound evenly on a torus of rectangular cross section. There are N turns of the blue wire in all. A red wire is thrown over the torus and is connected to a resistor, R, see Fig. 3. Find the magnitude and direction (clockwise or counterclockwise) of the current in the red wire, $I_{\text{red wire}}$.

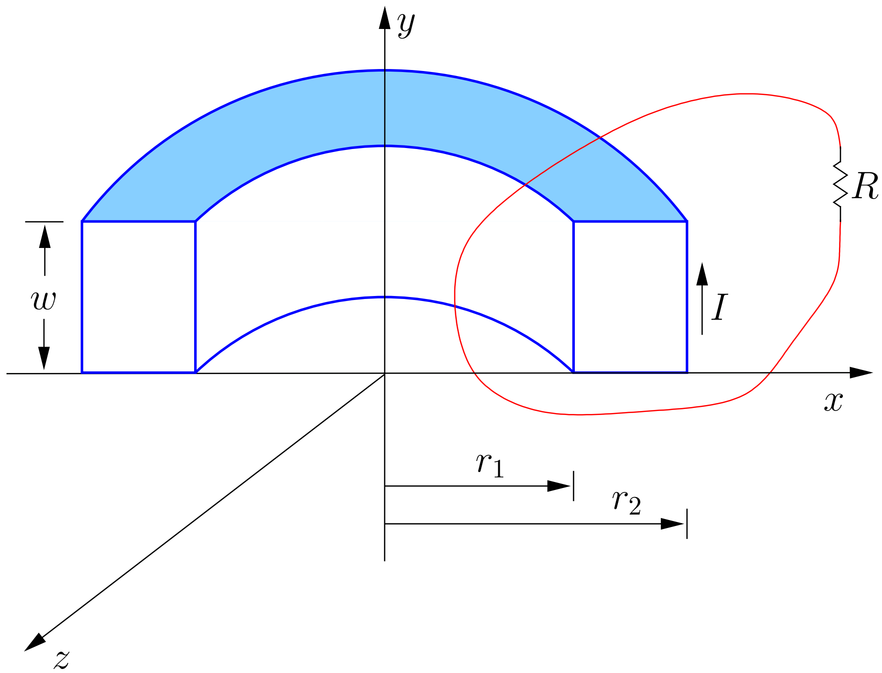

Figure 3: A blue wire carrying current $I = I_o t^3/3$ is wound evenly on a torus of rectangular cross section, with inner radius r_1 and outer radius r_2. There are N turns of the blue wire in all. A red wire is thrown over the torus and is connected to a resistor, R.

Solution: The magnetic field produced by the blue wire can be found via Ampere's Law,

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_o I_{\text{encl.}}, \tag{12}$$

where $I_{\text{encl.}}$ is the current enclosed by the Amperian loop. We choose the Amperian loop to be a circle of radius r centered at the origin (why?), see Fig. 4.

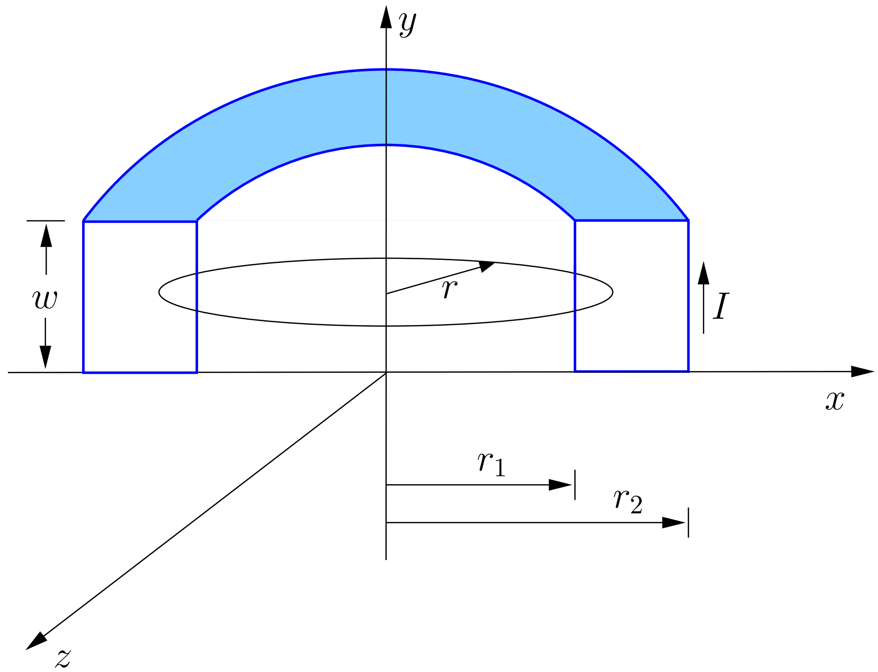

Figure 4: Amperian loop of radius r, laying "in the torus" is shown.

By symmetry, $\vec{B}$ is parallel to $\mathrm{d}\vec{\ell}$; thus, (12) yields

$$\oint B\mathrm{d}\ell = \mu_o I_{\text{encl.}}. \tag{13}$$

By symmetry, magnetic field B is constant on the Amperian loop; thus, (13) yields

$$\begin{aligned} B\oint \mathrm{d}\ell &= \mu_o I_{\text{encl.}} \\ B2\pi r &= \mu_o I_{\text{encl.}} \\ B &= \frac{\mu_o I_{\text{encl.}}}{2\pi r}, \end{aligned}$$

where $I_{\text{encl.}} = NI$. Thus, the magnetic field produced by the blue wire is given by

$$B = \frac{\mu_o NI}{2\pi r}. \tag{14}$$

The magnetic flux through the **area enclosed by the red wire** is given by

$$\begin{aligned} \Phi_B &= \int \vec{B}\cdot\mathrm{d}\vec{A} \\ &= \int_{r_1}^{r_2} B\,w\mathrm{d}r. \end{aligned} \tag{15}$$

Why are the limits of integration from r_1 to r_2 if we are calculating the magnetic flux through the **area enclosed by the red wire**? Substituting (14) into (15) and integrating yields

$$\Phi_B = \frac{\mu_o N I w}{2\pi} \ln\left(\frac{r_2}{r_1}\right). \tag{16}$$

The magnitude of the induced emf is given by

$$|\mathcal{E}| = \left|\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}\right|. \tag{17}$$

Substituting (16) into (17) yields

$$\begin{aligned} |\mathcal{E}| &= \frac{\mu_o N w}{2\pi} \ln\left(\frac{r_2}{r_1}\right) \left|\frac{\mathrm{d}I}{\mathrm{d}t}\right| \\ &= \frac{\mu_o N w}{2\pi} \ln\left(\frac{r_2}{r_1}\right) I_o t^2. \end{aligned} \tag{18}$$

Substituting (18) into Ohm's law yields

$$I_{\text{red wire}} = \frac{\mu_o N w I_o}{2\pi R} \ln\left(\frac{r_2}{r_1}\right) t^2,$$

where $I_{\text{red wire}}$ flows in the clockwise direction (why?).

3. Consider a conducting rod sitting on the top of an incline. The top of the incline is made from pair of frictionless conducting rails. There is a resistor, R, that connects the two rails, and a constant magnetic field directed vertically upwards with a magnitude B_o, see Fig. 5. The separation distance between the two frictionless conducting rails is L. If at time $t = 0$, the rod is released from rest, find the velocity of the rod as a function of time.

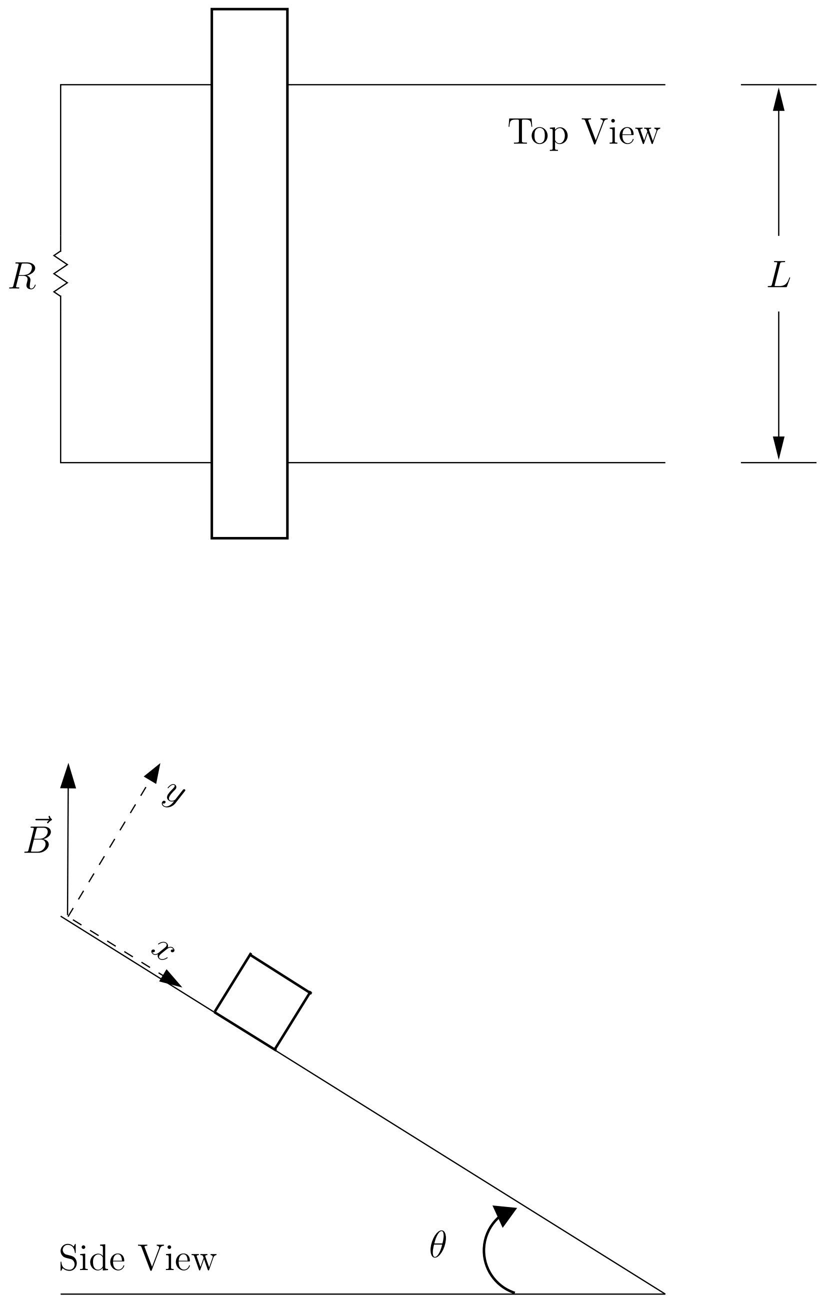

Figure 5: Top and side views of the conducting incline are shown. Notice that the pair of frictionless conducting rails, the conducting rod and the resistor form a complete circuit.

Solution: First, we write the magnetic field in terms of the given coordinate system, see Fig. 5,

$$\vec{B} = B_o \left(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\right). \tag{19}$$

The magnetic flux through the area enclosed by the resistor, rails and the rod is given by

$$\begin{aligned}
\Phi_B &= \int \vec{B}\cdot \mathrm{d}\vec{A} \\
&= \int_{x_o}^{x} \vec{B}\cdot\, L\mathrm{d}x'\,\hat{\boldsymbol{\jmath}} \\
&= \int_{x_o}^{x} B_o L\cos(\theta)\,\mathrm{d}x' \\
&= B_o L\cos(\theta)(x - x_o).
\end{aligned} \tag{20}$$

The magnitude of the induced emf is given by

$$|\mathcal{E}| = \left|\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}\right|. \tag{21}$$

Substituting (20) into (21) and identifying $\frac{\mathrm{d}x}{\mathrm{d}t}$ as velocity v yields

$$|\mathcal{E}| = B_o L\cos(\theta) v. \tag{22}$$

Substituting (22) into Ohm's law yields

$$I = \frac{B_o L\cos(\theta)}{R} v, \tag{23}$$

where I is the current in the rod flowing in the **positive** $\hat{\boldsymbol{k}}$ direction (why?). The magnetic force on the rod is given by

$$\begin{aligned}
\vec{F}_B &= \int I\mathrm{d}\vec{\ell}\times\vec{B} \\
&= \int_0^L I\mathrm{d}z\,\hat{\boldsymbol{k}}\times\left[B_o\left(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\right)\right], \quad \text{used (19) for } \vec{B} \\
&= -B_o I L\left(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}\right),
\end{aligned} \tag{24}$$

where I is given by (23). Writing the sum of all forces in the $\hat{\boldsymbol{\imath}}$ direction yields

$$-\frac{B_o^2 L^2\cos^2\theta}{R} v + mg\sin\theta = m\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{25}$$

We must solve (25) for v, so we rewrite (25) in the more manageable form

$$-av + b = \frac{\mathrm{d}v}{\mathrm{d}t}, \tag{26}$$

where $a = \frac{B_o^2 L^2 \cos^2\theta}{Rm}$ and $b = g\sin\theta$. Now, it is a trivial matter to find v, so we proceed without any comments.

$$\begin{aligned} -\int \mathrm{d}t &= \int \frac{\mathrm{d}v}{av - b} \\ -t + C &= \frac{\ln(av-b)}{a} \\ C_1 e^{-at} &= av - b \\ \frac{b}{a}\left(1 - e^{-at}\right) &= v. \end{aligned} \tag{27}$$

Finally, substituting a and b into (27) yields

$$v(t) = \frac{Rmg\sin\theta}{B_o^2 L^2 \cos^2\theta}\left(1 - e^{-\frac{B_o^2 L^2 \cos^2\theta}{Rm}t}\right). \tag{28}$$

It is interesting to plot (28) for some parameter values, see Fig. 6.

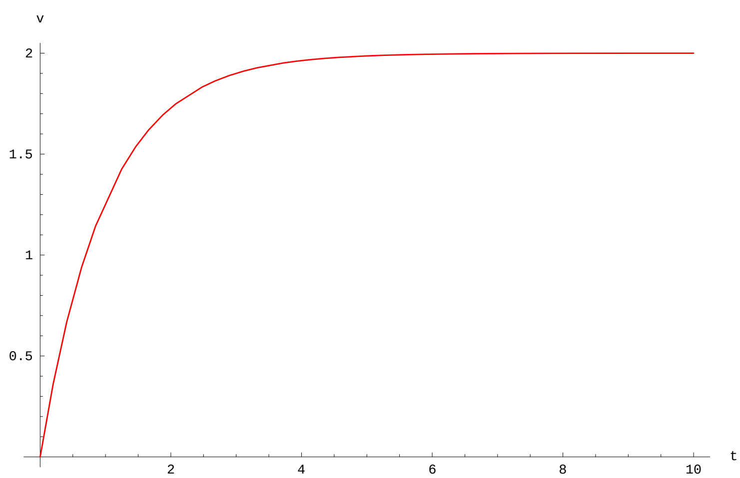

Figure 6: The velocity of the rod is shown for the following parameter values: $\frac{B_o^2 L^2 \cos^2\theta}{Rm} = 1$ and $g\sin\theta = 2$. Notice that the graph is flat for roughly $t > 5$. Can you explain this flat region physically?