1. Find the torque around the $x-$axis, $\vec{\tau}$, that the blue wire carrying a current I (see Fig. 1) experiences in the presence of a uniform magnetic field, $\vec{B} = B_o\hat{\boldsymbol{j}}$.

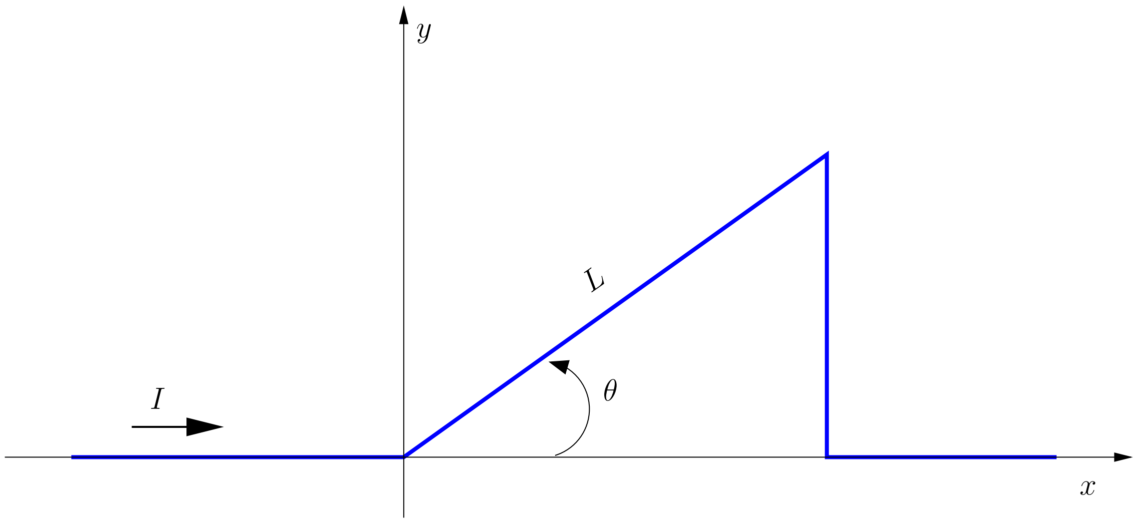

Figure 1: A wire carrying current I is shown in blue. L is the length of the "diagonal" part of the wire.

Solution: The differential form of the torque equation is given by (review Physics I)

$$\mathrm{d}\vec{\tau} = \vec{r} \times \mathrm{d}\vec{F}, \tag{1}$$

where $\vec{r}$ is the "lever arm" and $\mathrm{d}\vec{F}$ is the differential force that the wire experiences. $\mathrm{d}\vec{F}$ is the magnetic force; it's given by

$$\mathrm{d}\vec{F} = I\,\mathrm{d}\vec{\ell} \times \vec{B}. \tag{2}$$

The horizontal parts of the wire do not contribute to the torque because the "lever arm" is zero. The vertical part of the wire does not contribute to the torque because it experiences no force ($I\,\mathrm{d}\vec{\ell}$ is anti-parallel to $\vec{B}$, hence $I\,\mathrm{d}\vec{\ell} \times \vec{B} = 0$). Thus, we only have to worry about the "diagonal" part of the wire, i.e., $y = \tan(\theta)x$, where $\tan(\theta)$ is the slope. Computing $\mathrm{d}\vec{\ell}$ yields

$$\begin{aligned}
\vec{\ell} &= x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} \\
\vec{\ell} &= x\,\hat{\boldsymbol{\imath}} + \tan(\theta)x\,\hat{\boldsymbol{\jmath}} \\
\frac{\mathrm{d}\vec{\ell}}{\mathrm{d}x} &= 1\,\hat{\boldsymbol{\imath}} + \tan(\theta)\,\hat{\boldsymbol{\jmath}} \\
\mathrm{d}\vec{\ell} &= \mathrm{d}x\,\hat{\boldsymbol{\imath}} + \tan(\theta)\,\mathrm{d}x\,\hat{\boldsymbol{\jmath}}.
\end{aligned} \tag{3}$$

Substituting (3) into (2) yields

$$\mathrm{d}\vec{F} = IB_o\mathrm{d}x\,\hat{\boldsymbol{k}}. \tag{4}$$

Before we substitute (4) into (1), we need to know $\vec{r}$, the "lever arm" in (1). The "lever arm" is the vertical distance between the $x-$axis and the wire; it is given by

$$\begin{aligned} \vec{r} &= y\,\hat{\boldsymbol{\jmath}} \\ \vec{r} &= \tan(\theta)x\,\hat{\boldsymbol{\jmath}}. \end{aligned} \tag{5}$$

Finally, substituting (5) and (4) into (1) yields

$$\begin{aligned} \mathrm{d}\vec{\tau} &= IB_o\tan(\theta)x\mathrm{d}x\,\hat{\boldsymbol{\imath}} \\ \vec{\tau} &= IB_o\tan(\theta)\int_0^{L\cos(\theta)} x\,\mathrm{d}x\,\hat{\boldsymbol{\imath}} \\ \vec{\tau} &= \frac{IB_oL^2\sin(\theta)\cos(\theta)}{2}\,\hat{\boldsymbol{\imath}}. \end{aligned}$$

2. Find the net force on a wire bent into a semi-circle shape with radius R and carrying a current I (see Fig. 2) in the presence of a non-uniform magnetic field, $\vec{B} = \arctan(\theta)z\,\hat{\boldsymbol{k}}$. Make sure you evaluate any integrals that may arise!

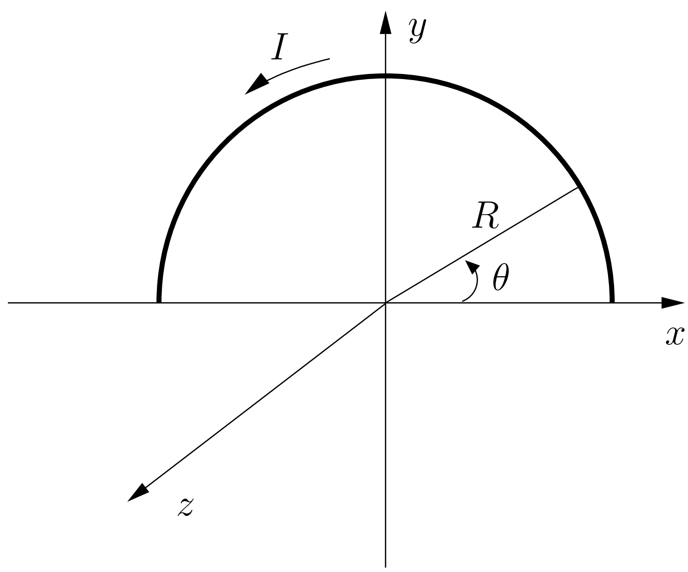

Figure 2: A semi-circular wire with radius R and current I lies in the $z = 0$ plane.

Solution: The magnetic force on the wire is given by

$$\mathrm{d}\vec{F} = I\,\mathrm{d}\vec{\ell}\times\vec{B}, \tag{6}$$

where $\mathrm{d}\vec{\ell}$ is computed in the "usual way", i.e.,

$$\begin{aligned}\vec{\ell} &= R\cos(\theta)\,\hat{\boldsymbol{\imath}} + R\sin(\theta)\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} \\ \frac{\mathrm{d}\vec{\ell}}{\mathrm{d}\theta} &= -R\sin(\theta)\,\hat{\boldsymbol{\imath}} + R\cos(\theta)\,\hat{\boldsymbol{\jmath}} \\ \mathrm{d}\vec{\ell} &= -R\sin(\theta)\,\mathrm{d}\theta\,\hat{\boldsymbol{\imath}} + R\cos(\theta)\,\mathrm{d}\theta\,\hat{\boldsymbol{\jmath}}. \end{aligned} \tag{7}$$

Substituting (7) into (6) yields

$$\begin{aligned}\mathrm{d}\vec{F} &= I\big[-R\sin(\theta)\,\mathrm{d}\theta\,\hat{\boldsymbol{\imath}} + R\cos(\theta)\,\mathrm{d}\theta\,\hat{\boldsymbol{\jmath}}\big] \times \vec{B} \\ \vec{F} &= \int_0^\pi I\big[-R\sin(\theta)\,\mathrm{d}\theta\,\hat{\boldsymbol{\imath}} + R\cos(\theta)\,\mathrm{d}\theta\,\hat{\boldsymbol{\jmath}}\big] \times \vec{B}. \end{aligned} \tag{8}$$

Now, we need to substitute $\vec{B}$, given by $\vec{B} = \arctan(\theta) z\,\hat{\boldsymbol{k}}$, into (8). However, (8) implicitly states that $\vec{B}$ must be evaluated on the wire (semi-circle). Evaluating $\vec{B}$ on the wire (semi-circle) yields

$$\begin{aligned}\vec{B} &= \arctan(\theta)\,0\,\hat{\boldsymbol{k}} \\ \vec{B} &= 0\,\hat{\boldsymbol{k}}. \end{aligned} \tag{9}$$

Finally, putting (9) into (8) yields

$$\vec{F} = 0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}.$$

3. Set up an expression for the net force on the wire (curve) $y = f(x)$, where $x_1 \le x \le x_2$ lies in the $z = 0$ plane and carries a current I (see Fig. 3). Assume that there is a non-uniform magnetic field $\vec{B} = \vec{B}(x, y, z)$ in the region of space where the wire lies.

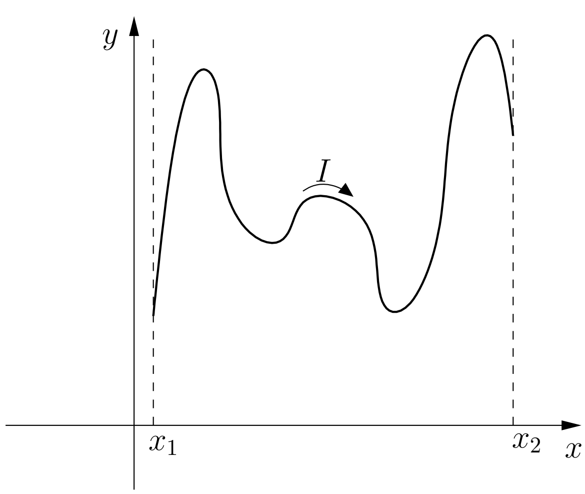

Figure 3: An arbitrary wire (curve), $y = f(x)$, carrying a current I (as indicated on the diagram) is shown.

Solution: The magnetic force on the wire is given by

$$\mathrm{d}\vec{F} = I\,\mathrm{d}\vec{\ell} \times \vec{B}, \tag{10}$$

where $\mathrm{d}\vec{\ell}$ is computed in the "usual way", i.e.,

$$\begin{aligned}
\vec{\ell} &= x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} \\
\vec{\ell} &= x\,\hat{\boldsymbol{\imath}} + f(x)\,\hat{\boldsymbol{\jmath}} \\
\frac{\mathrm{d}\vec{\ell}}{\mathrm{d}x} &= 1\,\hat{\boldsymbol{\imath}} + f'(x)\,\hat{\boldsymbol{\jmath}} \\
\mathrm{d}\vec{\ell} &= \mathrm{d}x\,\hat{\boldsymbol{\imath}} + f'(x)\,\mathrm{d}x\,\hat{\boldsymbol{\jmath}},
\end{aligned} \tag{11}$$

where $f'(x)$ denotes a derivative of $f(x)$ w.r.t x. Evaluating $\vec{B}$ on the wire (curve) yields

$$\begin{aligned}
\vec{B} &= \vec{B}(x, y, z) \\
\vec{B} &= \vec{B}(x, y = f(x), z = 0).
\end{aligned} \tag{12}$$

Finally, substituting (11) and (12) into (10) yields

$$\begin{aligned}
\mathrm{d}\vec{F} &= I\big[\mathrm{d}x\,\hat{\boldsymbol{\imath}} + f'(x)\,\mathrm{d}x\,\hat{\boldsymbol{\jmath}}\big] \times \vec{B}(x, y = f(x), z = 0) \\
\vec{F} &= \int_{x_1}^{x_2} I\big[\mathrm{d}x\,\hat{\boldsymbol{\imath}} + f'(x)\,\mathrm{d}x\,\hat{\boldsymbol{\jmath}}\big] \times \vec{B}(x, y = f(x), z = 0).
\end{aligned}$$

4. A particle with charge q is traveling with constant velocity $\vec{v}$, where $v \ll c$ and $c = \frac{1}{\sqrt{\epsilon_o \mu_o}}$ is the speed of light (we will learn this later in the course). Write an expression for the magnetic field, $\vec{B}$, that the particle generates, in terms of the electric field $\vec{E}$ and speed of light c.

Solution: The magnetic field due to a slow moving charge is **approximately** (here, "approximately" means we are ignoring retardation effects; don't worry if you don't know what this means) given by

$$\vec{B} = \frac{\mu_o}{4\pi}\frac{q\vec{v} \times \vec{r}}{r^3}, \tag{13}$$

and the electric field is given by

$$\vec{E} = \frac{q}{4\pi\epsilon_o}\frac{\vec{r}}{r^3}. \tag{14}$$

Solving (14) for $\frac{\vec{r}}{r^3}$ and substituting the result into (13) yields

$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2}. \tag{15}$$

Equation (15) is a "cute" equation relating the electric and magnetic fields. Later on in the course, we will learn that electric and magnetic fields are fundamentally related quantities; this is largely due to the work of James Clerk Maxwell, http://en.wikipedia.org/wiki/James_Clerk_Maxwell

5. Helmholtz coil. This problem is NOT on the quiz.

   (a) **Without using any symmetry arguments**, find the magnetic field anywhere on the $z-$axis.

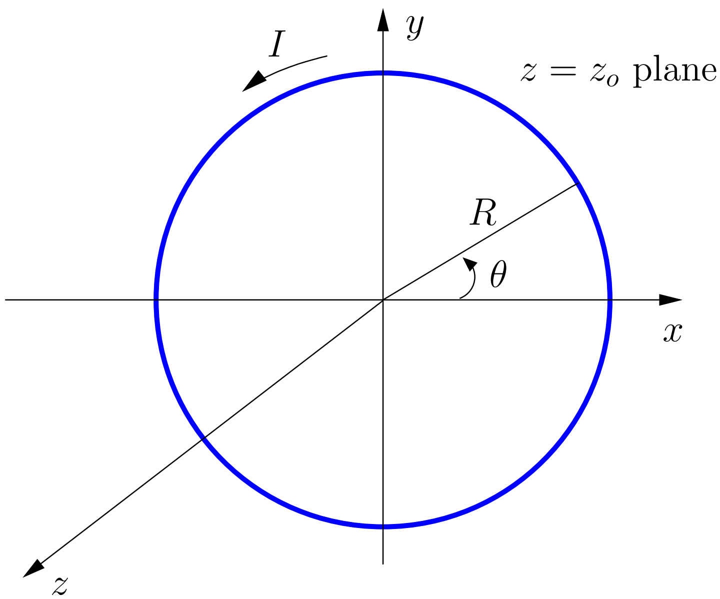

Figure 4: A circular wire with radius R and current I lies in the $z = z_o$ plane.

Solution: The magnetic field is given by

$$\vec{B} = \frac{\mu_o}{4\pi} \int \frac{I \, d\vec{\ell} \times \vec{r}}{r^3}, \tag{16}$$

where $\vec{r} = \vec{r}_f - \vec{r}_s$, and $d\vec{\ell}$ should really be $d\vec{r}_s$. Against all of my wishes, we will NOT use $d\vec{r}_s$ because your book and lecture notes call it $d\vec{\ell}$ and it's important to have uniform notation. Recall that $\vec{r}_s$ is the **source** variable and $\vec{r}_f$ is the **field** variable. Computing $\vec{r}_f$ yields

$$\vec{r}_f = z\,\hat{\boldsymbol{k}}, \tag{17}$$

since we want to know the magnetic field anywhere on the $z-$axis. Computing $\vec{r}_s$ yields

$$\vec{r}_s = R\cos(\theta)\,\hat{\boldsymbol{\imath}} + R\sin(\theta)\,\hat{\boldsymbol{\jmath}} + z_o\,\hat{\boldsymbol{k}}, \tag{18}$$

since the current loop that produces the field lies in the z_o-plane. Taking the difference between (17) and (18) yields

$$\vec{r} = -R\cos(\theta)\,\hat{\boldsymbol{\imath}} - R\sin(\theta)\,\hat{\boldsymbol{\jmath}} + (z - z_o)\,\hat{\boldsymbol{k}}. \tag{19}$$

$\mathrm{d}\vec{\ell}$ is computed in the "usual way", i.e.,

$$\begin{aligned}\vec{\ell} &= R\cos(\theta)\,\hat{\boldsymbol{\imath}} + R\sin(\theta)\,\hat{\boldsymbol{\jmath}} + z_o\,\hat{\boldsymbol{k}} \qquad \text{[compare this to (18)]}\\ \frac{\mathrm{d}\vec{\ell}}{\mathrm{d}\theta} &= -R\sin(\theta)\,\hat{\boldsymbol{\imath}} + R\cos(\theta)\,\hat{\boldsymbol{\jmath}}\\ \mathrm{d}\vec{\ell} &= -R\sin(\theta)\,\mathrm{d}\theta\,\hat{\boldsymbol{\imath}} + R\cos(\theta)\,\mathrm{d}\theta\,\hat{\boldsymbol{\jmath}}.\end{aligned} \tag{20}$$

Substituting (19) and (20) into (16) and integrating the x and y components of the field (recall that $\int_0^{2\pi}\sin(\theta)\,\mathrm{d}\theta = \int_0^{2\pi}\cos(\theta)\,\mathrm{d}\theta = 0$) yields

$$\vec{B} = \frac{\mu_o I R^2}{2\big[R^2 + (z - z_o)^2\big]^{3/2}}\,\hat{\boldsymbol{k}}. \tag{21}$$

You could have skipped the calculation of the x and y components of the field because both of them are zero, as can be shown by a symmetry argument. However, symmetry arguments are difficult to explain and possibly even more difficult to understand, hence the algebra-intensive exercise.

(b) Find the magnetic field due to two circular coils, each having a radius R and carrying current I in the same direction. Assume the center of the first circular coil is at $(0, 0, 0)$ and the center of the second circular coil is at $(0, 0, R)$. Furthermore, assume that N_1 is the number of wire loops in the first coil and N_2 is the number of wire loops in the second coil.

Solution: Using (21) with $z_o = 0$ yields the magnetic field produced by the first coil

$$\vec{B}_1 = \frac{N_1 \mu_o I R^2}{2\big[R^2 + z^2\big]^{3/2}}\,\hat{\boldsymbol{k}}. \tag{22}$$

Using (21) with $z_o = R$ yields the magnetic field produced by the second coil

$$\vec{B}_2 = \frac{N_2 \mu_o I R^2}{2\big[R^2 + (z - R)^2\big]^{3/2}}\,\hat{\boldsymbol{k}}. \tag{23}$$

The total magnetic field is given by

$$\vec{B} = \frac{N_1 \mu_o I R^2}{2\big[R^2 + z^2\big]^{3/2}}\,\hat{\boldsymbol{k}} + \frac{N_2 \mu_o I R^2}{2\big[R^2 + (z - R)^2\big]^{3/2}}\,\hat{\boldsymbol{k}}$$

This arrangement of coils is known as the Helmholtz coil,
http://en.wikipedia.org/wiki/Helmholtz_coil and produces a surprisingly uniform magnetic field on the $z-$axis (see Fig. 5).

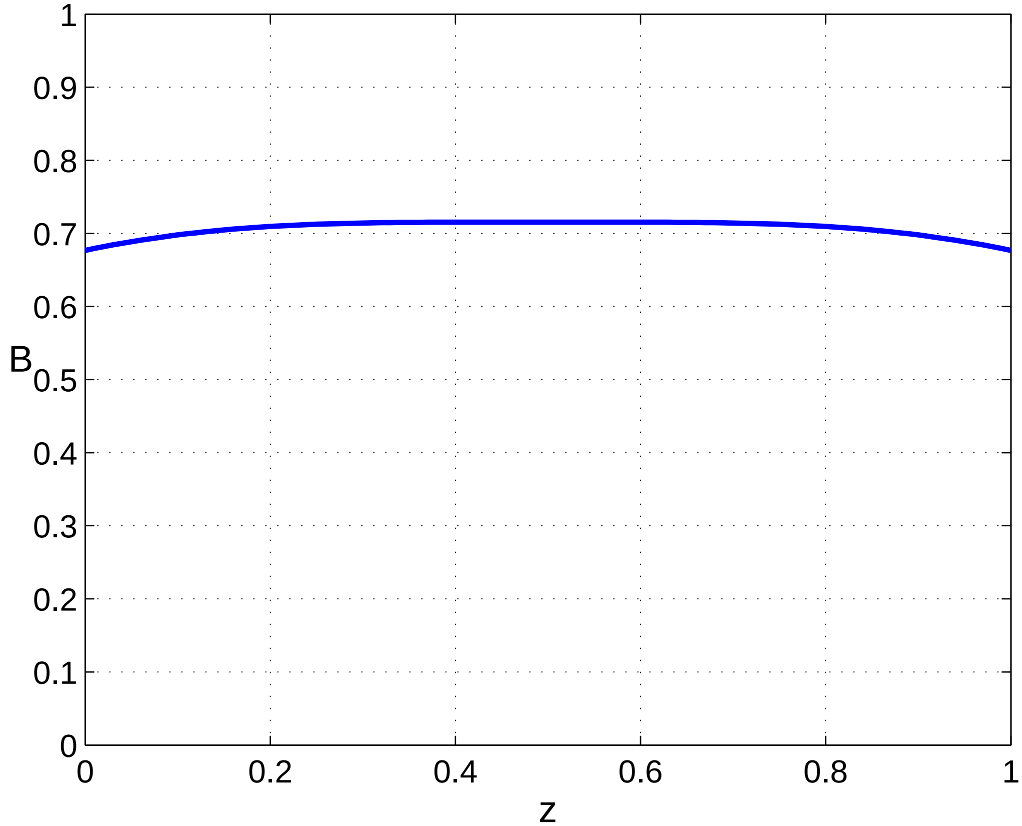

Figure 5: $N_1\mu_o I = N_2\mu_o I = 1$ and $R = 1$.