1. The space between two concentric metal spheres with radii a and b is filled with a material of resistivity ρ, see Fig. (1). A potential difference is set up between the inner and the outer spheres so that current flows radially outward from the inner sphere. Find the resistance of this arrangement.

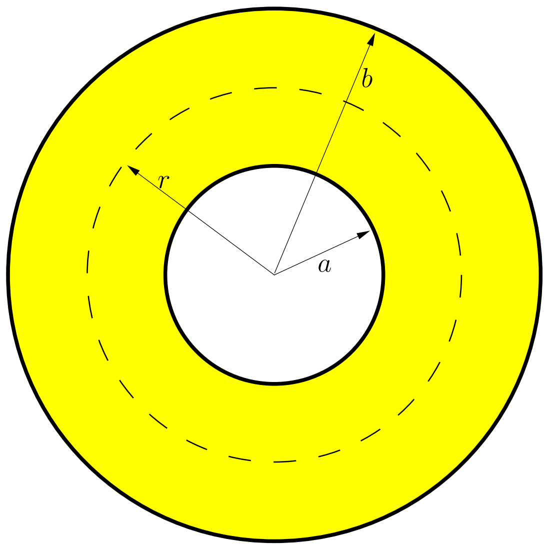

Figure 1: Two concentric spheres are shown in a cross-section view.

Solution: For a uniform wire, resistance is given by

$$R = \rho \frac{L}{A}, \tag{1}$$

where L is the length of the wire and A is the cross-sectional area that the current flows through. We need to adapt (1) for our situation. In our situation, area changes as current flows from the inner sphere to the outer sphere, thus, we must integrate. The area is given by

$$A = 4\pi r^2, \tag{2}$$

and the current flows a differential distance $\mathrm{d}r$. Using (1) and (2) yields

$$R = \rho \int_a^b \frac{\mathrm{d}r}{4\pi r^2}. \tag{3}$$

Integrating (3) and simplifying yields

$$R = \frac{\rho}{4\pi}\left(\frac{b-a}{ab}\right).$$

2. A resistor circuit is shown in Fig. (2). Please follow the directions given in the figure.

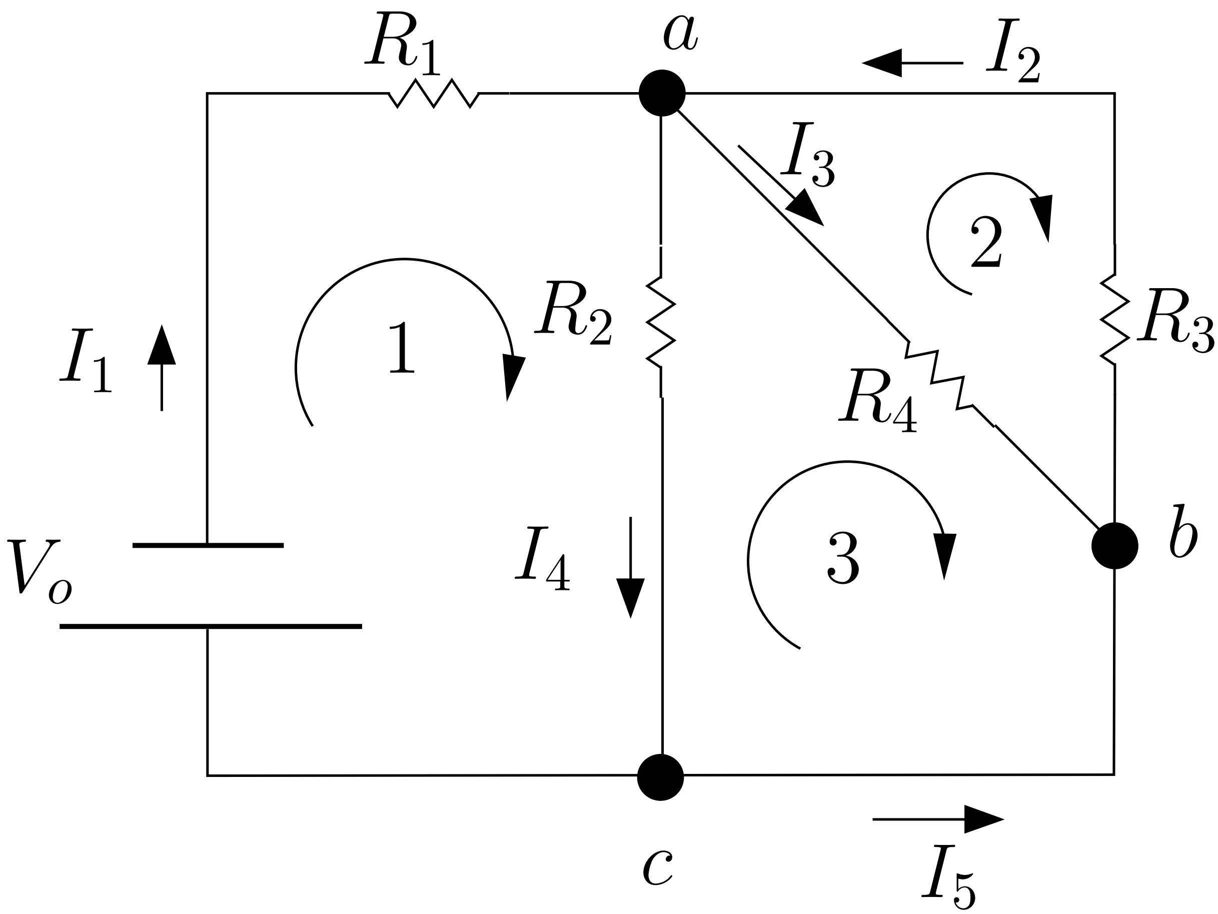

Figure 2: A resistor network is shown.

(a) Find R_{eq}.

Solution:

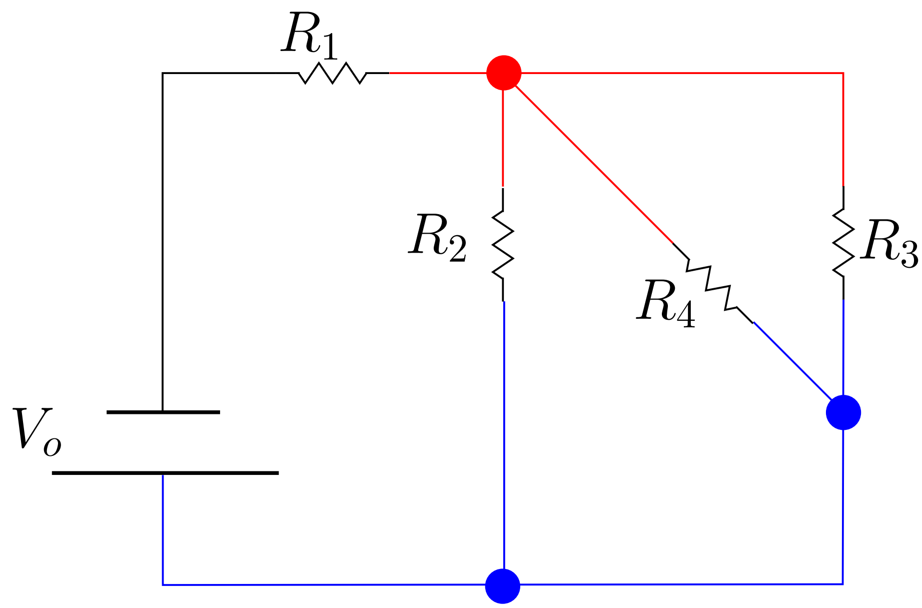

Figure 3: First, we color-code our voltages, which helps us recognize patterns.

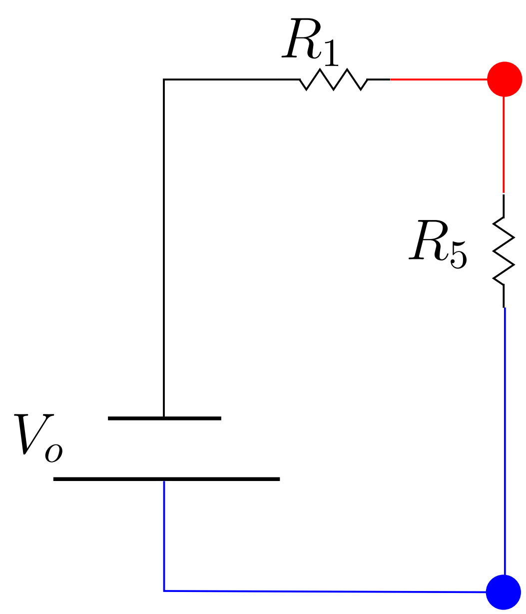

Figure 4: $R_5 = R_2 \| R_3 \| R_4$

Thus,

$$R_{\text{eq}} = R_1 \perp R_5,$$

where $R_5 = R_2 \| R_3 \| R_4$

(b) Write down enough equations so you can solve for all unknown currents shown in Fig. (2). Do NOT solve the equations.

Solution: Applying Kirchhoff's loop rule to loop 1 (see Fig. (2)) yields

$$-V_o - R_1 I_1 - R_2 I_4 = 0.$$

Applying Kirchhoff's loop rule to loop 2 (see Fig. (2)) yields

$$+R_3 I_2 + R_4 I_3 = 0.$$

Applying Kirchhoff's loop rule to loop 3 (see Fig. (2)) yields

$$+R_2 I_4 - R_4 I_3 = 0.$$

Applying Kirchhoff's junction rule at a (see Fig. (2)) yields

$$I_1 + I_2 = I_3 + I_4.$$

Applying Kirchhoff's junction rule at b (see Fig. (2)) yields

$$I_3 + I_5 = I_2.$$

We already have 5 equations, so we, should be able to find all 5 unknown currents. If you want, you can apply Kirchhoff's junction rule at c (see Fig. (2)) to obtain

$$I_4 = I_1 + I_5,$$

but this equation is redundant, i.e., you are working harder than you have to. Notice the general pattern; first we apply Kirchhoff's loop rule to all "inner loops/circuits", then we apply Kirchhoff's junction rule to obtain as many equations as there are unknowns.

3. An RC circuit is shown in Fig. (5). The capacitor is initially uncharged and at time $t = 0$, the switch is closed.

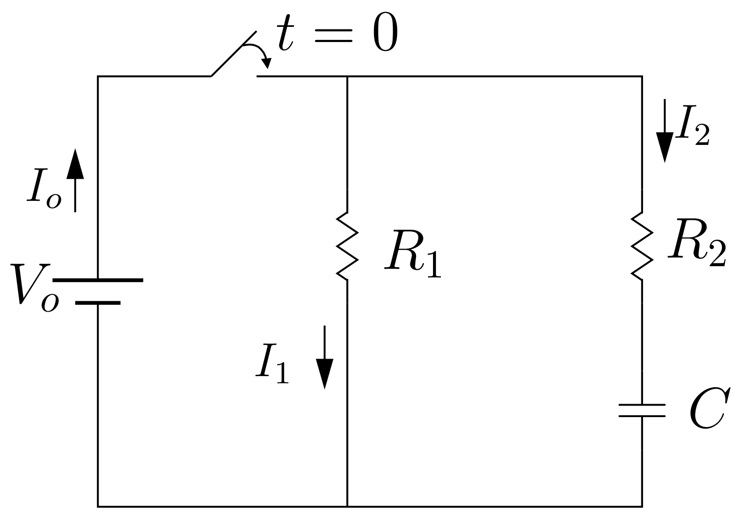

Figure 5: The switch is closed at time $t = 0$ and the capacitor is initially uncharged.

(a) Starting with Kirchhoff's rules, derive an expression for I_1.

> **Solution:** Applying Kirchhoff's loop rule (see Fig. (5)) yields
>
> $$V_o - R_1 I_1 = 0$$
> $$I_1 = \frac{V_o}{R_1}. \tag{4}$$

(b) Starting with Kirchhoff's rules, derive an expression for I_2.

> **Solution:** Applying Kirchhoff's loop rule (see Fig. (5)) yields
>
> $$V_o - R_2 I_2 - \frac{Q}{C} = 0. \tag{5}$$

Taking a derivative w.r.t time of both sides of (5) yields

$$\underbrace{\frac{\mathrm{d}V_0}{\mathrm{d}t}}_{=0} - R_2\frac{\mathrm{d}I_2}{\mathrm{d}t} - \frac{1}{C}\underbrace{\frac{\mathrm{d}Q}{\mathrm{d}t}}_{=I_2} = \underbrace{\frac{\mathrm{d}0}{\mathrm{d}t}}_{=0}$$

$$-R_2\frac{\mathrm{d}I_2}{\mathrm{d}t} - \frac{I_2}{C} = 0$$

$$\frac{\mathrm{d}I_2}{I_2} = -\frac{\mathrm{d}t}{R_2C}$$

$$\int\frac{\mathrm{d}I_2}{I_2} = -\int\frac{\mathrm{d}t}{R_2C}$$

$$\ln I_2 = -\frac{t}{R_2C} + \underbrace{K_1}_{\text{constant}}$$

$$I_2 = K_2e^{-\frac{t}{R_2C}}, \tag{6}$$

where K_2 is some other constant. To find K_2, we must apply the initial condition, i.e., what was happening in the circuit at the instant the switch was closed. At the instant the switch was closed, the capacitor was just beginning to charge, thereby, acting like an ideal wire. Therefore, at $t = 0$, $I_2 = \frac{V_o}{R_2}$. Applying this initial condition to (6) yields

$$I_2\left(t=0\right) = \frac{V_o}{R_2}$$

$$K_2e^{-\frac{0}{R_2C}} = \frac{V_o}{R_2}$$

$$K_2 = \frac{V_o}{R_2}. \tag{7}$$

Finally, substituting (7) into (6) yields

$$I_2 = \frac{V_o}{R_2}e^{-\frac{t}{R_2C}}. \tag{8}$$

(c) Starting with Kirchhoff's rules, derive an expression for I_o.

Solution: Applying Kirchhoff's junction rule (see Fig. (5)), then substituting (4) and (8) yields

$$I_o = I_1 + I_2$$

$$I_o = \frac{V_o}{R_1} + \frac{V_o}{R_2}e^{-\frac{t}{R_2C}}.$$