1. We want to design a spherical vacuum capacitor of a given radius a for the outer sphere, which will be able to store the greatest amount of electrical energy, subject to the constraint that the electric field strength at the surface of the inner sphere may not exceed E_o.

   (a) What radius b should be chosen for the inner spherical conductor?

Solution: Let Q be the amount of positive charge on the inner sphere; the electric field between the two spheres is given by (use Gauss's Law to show this or look at the Recitation 2 problems)

$$E = \frac{Q}{4\pi\epsilon_o r^2}, \quad b < r < a, \tag{1}$$

where r is the radius drawn from the origin of the coordinate system. The electric field must not exceed E_o at b, thus, letting $E = E_o$, $r = b$ in (1), and solving for Q yields

$$Q = 4\pi\epsilon_o b^2 E_o. \tag{2}$$

The electric potential difference, $V_b - V_a$, is given by (see Recitation 2 problems)

$$\begin{aligned} V_b - V_a &= \int_b^a \vec{E} \cdot \mathrm{d}\vec{\ell} \\ &= \int_b^a \vec{E} \cdot \mathrm{d}x\hat{\boldsymbol{\imath}} \\ &= \int_b^a E\mathrm{d}x \\ &= \frac{Q}{4\pi\epsilon_o}\int_b^a \frac{1}{x^2}\mathrm{d}x \\ &= \frac{Q(a-b)}{4\pi\epsilon_o ab}, \end{aligned} \tag{3}$$

where Q is given by (2). The amount of electrical energy stored in a capacitor is

$$\begin{aligned} U &= \frac{1}{2}Q\triangle V \\ &= \frac{1}{2}Q(V_b - V_a) \\ &= \frac{1}{2}\left(4\pi\epsilon_o b^2 E_o\right)\left(\frac{Q(a-b)}{4\pi\epsilon_o ab}\right) \\ &= \frac{4\pi\epsilon_o}{2a}E_o^2 b^3(a-b). \end{aligned} \tag{4}$$

We want to find b that maximizes U; thus, the derivative of U with respect to (w.r.t) b must vanish. Differentiating (4) w.r.t b yields

$$b = \frac{3}{4}a. \tag{5}$$

(b) How much energy can be stored?

Solution: Substituting (5) into (4) yields

$$U = 4\pi\epsilon_o \frac{27a^3E_o^2}{512}.$$

2. An experimentalist observes a particle of charge q, mass m, and energy qV_o moving in a circular orbit of radius r_o inside a cylindrical capacitor with inner radius a and outer radius b, see Fig. (1).

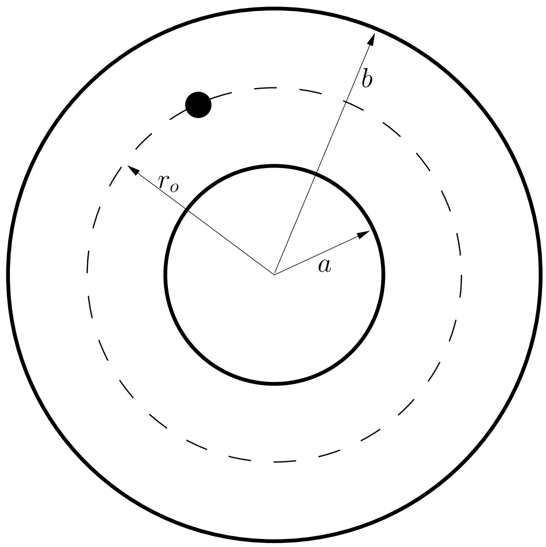

Figure 1: A particle of charge q, mass m, and energy qV_o moving inside a cylindrical capacitor is shown.

(a) Assume the cylindrical capacitor is very long compared to the space between its walls, i.e., $\frac{b-a}{\ell} \ll 1$. Furthermore, assume that the moving particle has no effect on the cylindrical capacitor. Find the capacitance of the cylindrical capacitor.

Solution: Let Q be the amount of positive charge on the inner cylinder; the electric field between the two cylinders is given by (use Gauss's Law to show this or look at the recitation two problems)

$$E = \frac{Q}{2\pi\ell\epsilon_o r}, \quad a < r < b, \tag{6}$$

The electric potential difference, $V_a - V_b$, is given by (see Recitation 2 problems)

$$\begin{aligned} V_a - V_b &= \int_a^b \vec{E} \cdot \mathrm{d}\vec{\ell} \\ &= \int_a^b \vec{E} \cdot \mathrm{d}x\hat{\boldsymbol{\imath}} \\ &= \int_a^b E \mathrm{d}x \\ &= \frac{Q}{2\pi\ell\epsilon_o} \int_a^b \frac{1}{x}\,\mathrm{d}x \\ &= \frac{Q}{2\pi\ell\epsilon_o} \ln\left(\frac{b}{a}\right). \end{aligned}$$

The capacitance is given by

$$\begin{aligned} C &= \frac{Q}{\triangle V} \\ &= \frac{Q}{V_a - V_b} \\ &= \frac{2\pi\ell\epsilon_o}{\ln(b/a)} \end{aligned} \tag{7}$$

(b) The experimentalist quickly observes that the particle only moves in a circular orbit with radius r_o when $V_a - V_b$ has a very special value. Find this value. (No, it's not zero.)

Solution: The electric field between the walls of the capacitor is (see Recitation 2 problems)

$$E = \frac{Q}{2\pi\ell\epsilon_o r}. \tag{8}$$

Substituting $C = \frac{Q}{V_a - V_b}$ into (8) and using (7) yields

$$E = \frac{V_a - V_b}{r\ln(b/a)}. \tag{9}$$

Setting the particle's kinetic energy equal to qV_o yields

$$v^2 = \frac{2qV_o}{m}, \tag{10}$$

where v is the particle's speed. Finally, using (9), (10), and Newton's Second Law yields

$$\begin{aligned} q\frac{V_a - V_b}{r_o \ln(b/a)} &= m\frac{v^2}{r_o} \\ V_a - V_b &= 2V_o \ln\left(\frac{b}{a}\right). \end{aligned}$$

3. Find the equivalent capacitance, C_{eq}, of the circuit shown in Fig. (2). Data: $C_1 = 1, C_2 = 2, C_3 = 3, C_4 = 4, C_5 = 5, C_6 = 6, C_7 = 7$.

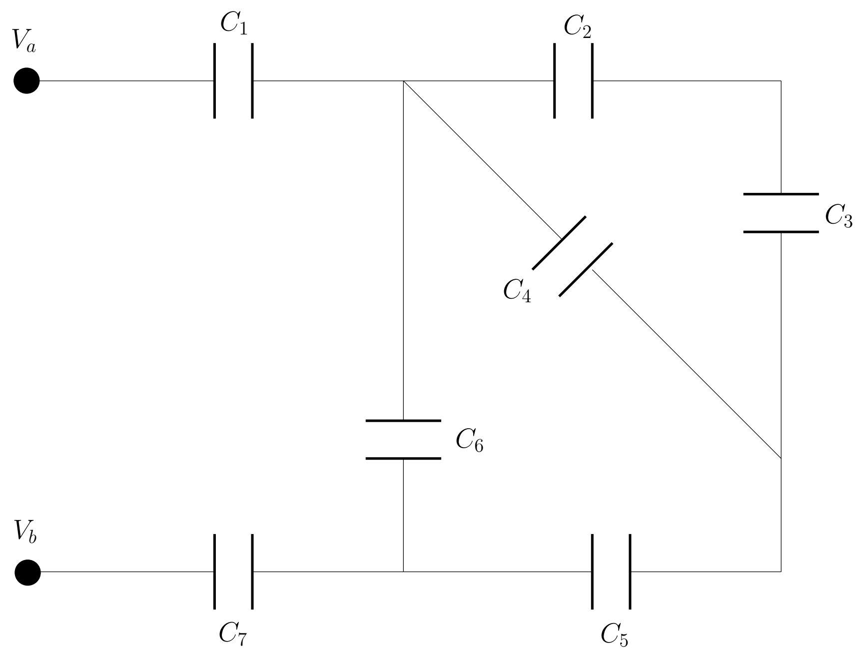

Figure 2: Find C_{eq} for this circuit. Hint: use color pencils/pens.

Solution:

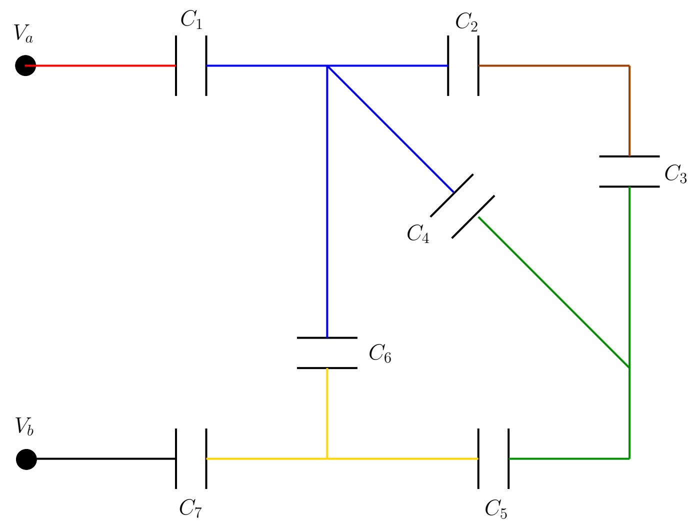

Figure 3: Capacitors in series are denoted by $\perp$, and capacitors in parallel are denoted by $\|$.

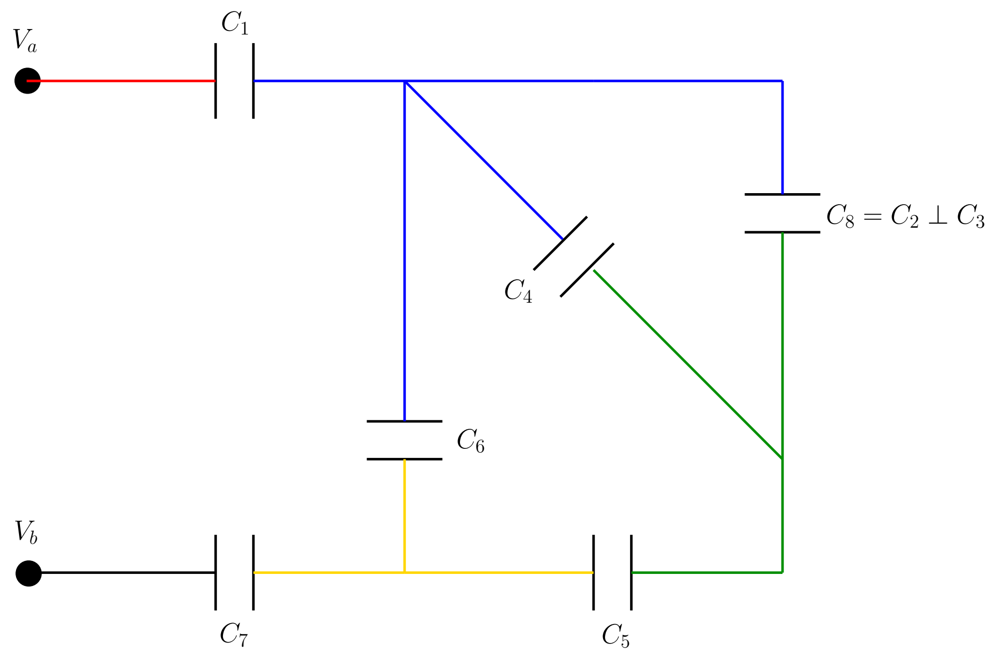

Figure 4: $C_8 = C_2 \perp C_3$

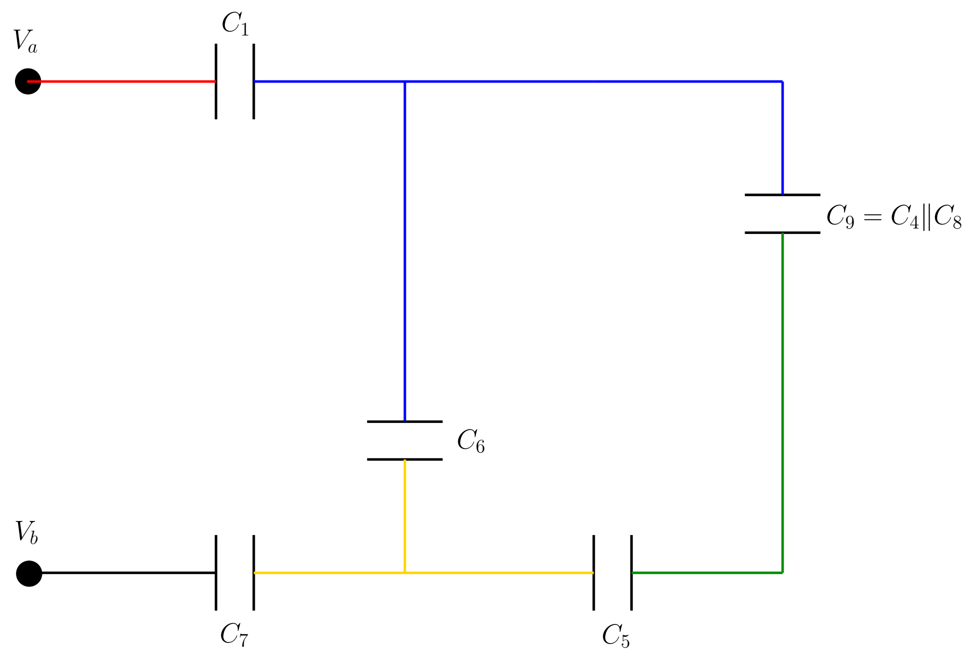

Figure 5: $C_9 = C_4 \| C_8$

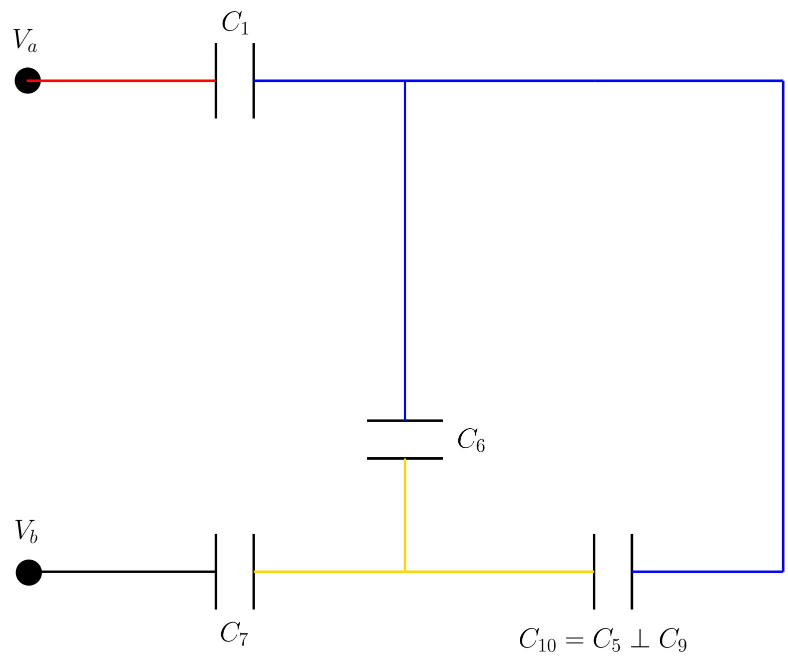

Figure 6: $C_{10} = C_5 \perp C_9$

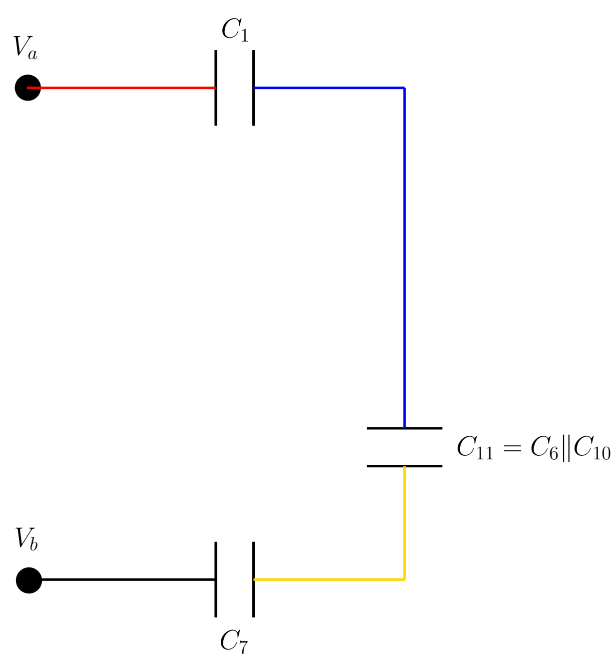

Figure 7: $C_{11} = C_6 \| C_{10}$

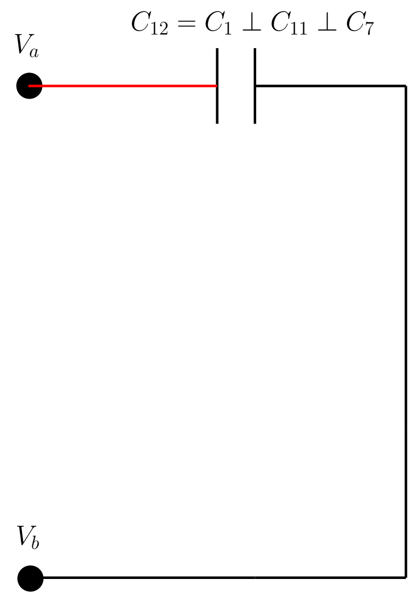

Figure 8: $C_{12} = C_1 \perp C_{10} \perp C_7$

Recall that capacitors in series obey

$$C_{\text{eq}} = \left(\sum_{i=1}^{N} C_i^{-1} \right)^{-1}, \tag{11}$$

and capacitors in parallel obey

$$C_{\text{eq}} = \sum_{i=1}^{N} C_i. \tag{12}$$

Using Fig. (3)-(8), (11), and (12) yields

$$C_{\text{eq}} = \frac{3052}{3845} \approx 0.794$$

4. Prove that a network of capacitors **cannot** always be grouped into series and parallel combinations.

Solution: We prove the above by finding a counter example, see Fig. 9.

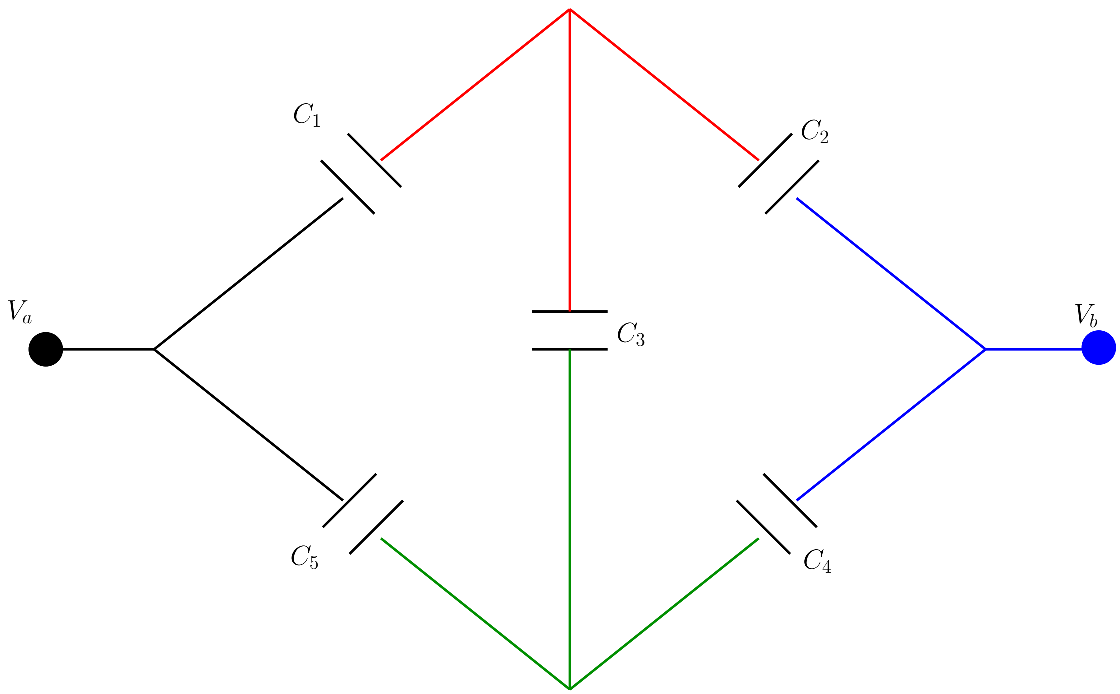

Figure 9: The above circuit cannot be reduced by grouping capacitors into series and parallel combinations.