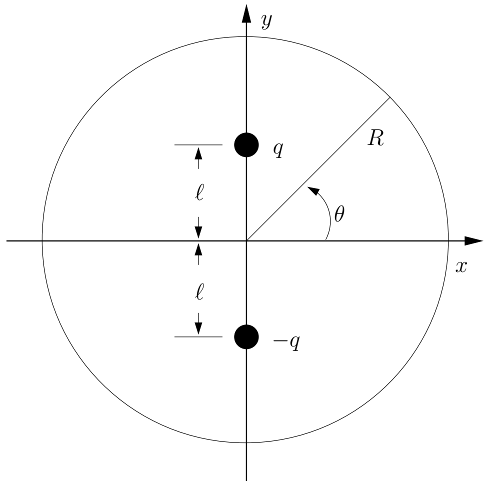

Figure 1: For the above electric dipole, $p = 2\ell q$.

1. Dipole in a 2-D world because the 3-D world is too damn hard!
   (a) Find the electric potential anywhere in the xy-plane (see Fig. 1).

Solution: The electric potential due to N point particles is given by

$$V = \frac{1}{4\pi\epsilon_o}\sum_{i=1}^{N}\frac{q_i}{r_i}, \quad \text{where } \vec{r} = \vec{r}_f - \vec{r}_{s_i}. \tag{1}$$

$\vec{r}_f$ is called the **field point** and $\vec{r}_{s_i}$ is the i^{th} **source point**.
We will call the positive charge the first charge and the negative charge the second charge. Thus,

$$\vec{r}_f = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \qquad \vec{r}_{s_1} = \ell\hat{\boldsymbol{\jmath}} \qquad \vec{r}_{s_2} = -\ell\hat{\boldsymbol{\jmath}},$$

Computing all quantities needed for (1) yields

$$\begin{aligned} \vec{r}_1 &= \vec{r}_f - \vec{r}_{s_1} & \vec{r}_2 &= \vec{r}_f - \vec{r}_{s_2} \\ &= x\hat{\boldsymbol{\imath}} + (y-\ell)\hat{\boldsymbol{\jmath}} & &= x\hat{\boldsymbol{\imath}} + (y+\ell)\hat{\boldsymbol{\jmath}} \\ r_1 &= ||\vec{r}_f - \vec{r}_{s_1}|| & r_2 &= ||\vec{r}_f - \vec{r}_{s_2}|| \\ &= \sqrt{x^2 + (y-\ell)^2} & &= \sqrt{x^2 + (y+\ell)^2}. \end{aligned} \tag{2}$$

Finally, substituting (2) into (1) yields

$$V = \frac{q}{4\pi\epsilon_o}\left[\frac{1}{\sqrt{x^2 + (y-\ell)^2}} - \frac{1}{\sqrt{x^2 + (y+\ell)^2}}\right] \tag{3}$$

(b) Find the $x-$component of the electric field from the expression you have obtained for the electric potential in part (a). Is your result consistent with what you obtained during the **first** recitation?

Solution: The relationship between an electric field and electric potential is given by

$$\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial V}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial V}{\partial z}\hat{\boldsymbol{k}}\right]. \tag{4}$$

We are only interested in the x component of the electric field, thus, (4) reduces to

$$E_x = -\frac{\partial V}{\partial x}, \tag{5}$$

where E_x denotes the $x-$component of the electric field. Substituting (3) into (5) and computing the partial derivative yields

$$E_x = \frac{q}{4\pi\epsilon_o}\left[\frac{x}{[x^2+(y-\ell)^2]^{3/2}} - \frac{x}{[x^2+(y+\ell)^2]^{3/2}}\right].$$

The above equation is consistent with the result of first recitation. If it is not, check your algebra!

2. Given a curve $y = f(x)$ where $x_1 \leq x \leq x_2$ in the $xy-$plane, find the electric potential at some field point, (a, b). Assume that the curve has a linear charge density given by $\lambda = g(x)$.

Solution: The electric potential is given by

$$V = \frac{1}{4\pi\epsilon_o}\int \frac{\mathrm{d}q}{r}, \tag{6}$$

where $\vec{r} = \vec{r}_f - \vec{r}_s$. First we find $\vec{r}_f$ and $\vec{r}_s$ as follows:

$$\begin{aligned}\vec{r}_f &= a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}\\ \vec{r}_s &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}\\ &= x\hat{\boldsymbol{\imath}} + f(x)\hat{\boldsymbol{\jmath}} \quad \text{recall that } y = f(x)\end{aligned}$$

Thus,

$$\begin{aligned}\vec{r} &= [a-x]\,\hat{\boldsymbol{\imath}} + [b-f(x)]\,\hat{\boldsymbol{\jmath}}\\ r &= \sqrt{[a-x]^2 + [b-f(x)]^2}.\end{aligned} \tag{7}$$

dq is given by $\mathrm{d}q = \lambda \mathrm{d}r_s$ and can be found in the usual manner,

$$\begin{aligned}
\vec{r}_s &= x\hat{\imath} + f(x)\hat{\jmath} \\
\frac{\mathrm{d}\vec{r}_s}{\mathrm{d}x} &= 1\hat{\imath} + f'(x)\hat{\jmath} \\
\frac{\mathrm{d}r_s}{\mathrm{d}x} &= \left\| \frac{\mathrm{d}\vec{r}_s}{\mathrm{d}x} \right\| \\
&= \sqrt{1 + [f'(x)]^2} \\
\mathrm{d}r_s &= \sqrt{1 + [f'(x)]^2}\mathrm{d}x. \qquad (8)
\end{aligned}$$

Note that (8) is just an arc length formula from Calculus. Finally, substituting (7) and (8) into (6) yields

$$V = \frac{1}{4\pi\epsilon_o} \int_{x_1}^{x_2} \frac{\lambda}{\sqrt{(a-x)^2 + (b - f(x))^2}} \sqrt{1 + [f'(x)]^2}\, \mathrm{d}x \qquad (9)$$

where $\lambda = g(x)$.

3. A sphere with radius R and volume charge density $\rho = \rho_o r^n$, where $\rho_o > 0$ and $n \geq 0$, is centered at the origin of a coordinate system.

(a) Find the electric field inside the sphere, i.e., $r < R$.

Solution: Due to the symmetry of the problem, we use Gauss's Law, which is given by

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\text{encl.}}}{\epsilon_o}. \qquad (10)$$

We want to find the electric field inside the sphere, so, we choose the Gaussian surface to be a sphere with radius $r < R$ (why?) centered at the origin of the coordinate system. The volume charge density is spherically symmetrical, and since $\rho_o > 0$, the electric field points radially outward (if ρ_o was negative then $\vec{E}$ would point radially inward). Effectively, this symmetry argument allows us to compute the dot product in (10) to yield

$$\oint E \mathrm{d}A = \frac{Q_{\text{encl.}}}{\epsilon_o}. \qquad (11)$$

The electric field, E, in (11) is an unknown function, but it must be only a function of r because volume charge density only depends on r. This implies that the electric field is constant on the Gaussian surface (here we are thinking of the Gaussian surface as being fixed at some r), thus we can bring E outside the integral to yield

$$\begin{aligned}
E \oint \mathrm{d}A &= \frac{Q_{\text{encl.}}}{\epsilon_o} \\
E 4\pi r^2 &= \frac{Q_{\text{encl.}}}{\epsilon_o}. \qquad (12)
\end{aligned}$$

Now, we must compute the right hand side (RHS) of (12), where $Q_{\text{encl.}}$ is the amount of charge enclosed by the Gaussian surface with radius r. We recognize that we can find $Q_{\text{encl.}}$ by integrating ρ from 0 to r using a mathematical technique commonly referred to as "integration by spherical shells". A surface area of one spherical shell of radius ζ is $4\pi\zeta^2$, thus,

$$\begin{aligned} Q_{\text{encl.}} &= \int_0^r \rho \, \text{dVolume} \\ &= \int_0^r \rho \, 4\pi\zeta^2 \text{d}\zeta \\ &= \int_0^r \rho_o \zeta^n \, 4\pi\zeta^2 \text{d}\zeta \\ &= 4\pi\rho_o \frac{r^{n+3}}{n+3} \end{aligned} \tag{13}$$

Notice that ζ was just a dummy integration variable; we could NOT use r because that is our limit of integration (annoying rule from Calculus!). Finally, substituting (13) into (12) and simplifying yields

$$E = \frac{\rho_o r^{n+1}}{(n+3)\epsilon_o}, \quad r < R. \tag{14}$$

Notice that (14) is only valid inside the sphere, i.e., $r < R$.

(b) Find the electric field outside the sphere, i.e., $r > R$.

Solution: We can find the electric field outside the sphere, i.e., $r > R$, as in part (a) but we have to be more careful when we compute $Q_{\text{encl.}}$ since our Gaussian surface lies outside the physical sphere, and $\rho = 0$ for $r > R$. In other words,

$$\begin{aligned} Q_{\text{encl.}} &= \int_0^r \rho \, \text{dVolume} \\ &= \int_0^R \rho \, 4\pi\zeta^2 \text{d}\zeta + \int_R^r \rho \, 4\pi\zeta^2 \text{d}\zeta \\ &= \int_0^R \rho_o \zeta^n \, 4\pi\zeta^2 \text{d}\zeta + \int_R^r 0 \, 4\pi\zeta^2 \text{d}\zeta \\ &= 4\pi\rho_o \frac{R^{n+3}}{n+3}. \end{aligned} \tag{15}$$

Finally, substituting (15) into (12) and simplifying yields

$$E = \frac{\rho_o R^{n+3}}{r^2(n+3)\epsilon_o}, \quad r > R. \tag{16}$$

Notice that (16) is only valid outside the sphere, i.e., $r > R$.

(c) Find the electric potential difference between the center and surface of the sphere, i.e., $V_{\text{center}} - V_{\text{surface}}$.

Solution: The electric potential at some point A is given by (compare with (4))

$$V_A = -\int_{\text{ref.}}^{A} \vec{E}\cdot\, d\vec{\ell}, \tag{17}$$

where $d\vec{\ell} = dx\hat{\boldsymbol{\imath}} + dy\hat{\boldsymbol{\jmath}} + dz\hat{\boldsymbol{k}}$ and ref. denotes a reference point. Notice the annoying negative sign in (17); this sign comes from a convention that will cause you lots of pain. (This convention is used by physicists almost universally; however, mathematicians don't usually put any negative signs when defining a scalar field. What misery!!!) Here, we have eliminated the pain by giving you an explicit expression for $d\vec{\ell}$. There is also a fair bit of pain associated with the reference point, but we will avoid it by always talking about electric potential difference between two well-defined points, hence, $V_{\text{center}} - V_{\text{surface}}$. Here is how you may dispose of the reference point:

$$\begin{aligned}
V_{\text{center}} &= -\int_{\text{ref.}}^{\text{center}} \vec{E}\cdot\, d\vec{\ell} \\
V_{\text{surface}} &= -\int_{\text{ref.}}^{\text{surface}} \vec{E}\cdot\, d\vec{\ell} \\
V_{\text{center}} - V_{\text{surface}} &= -\int_{\text{ref.}}^{\text{center}} \vec{E}\cdot\, d\vec{\ell} + \int_{\text{ref.}}^{\text{surface}} \vec{E}\cdot\, d\vec{\ell} \\
V_{\text{center}} - V_{\text{surface}} &= \int_{\text{center}}^{\text{ref.}} \vec{E}\cdot\, d\vec{\ell} + \int_{\text{ref.}}^{\text{surface}} \vec{E}\cdot\, d\vec{\ell} \\
V_{\text{center}} - V_{\text{surface}} &= \int_{\text{center}}^{\text{surface}} \vec{E}\cdot\, d\vec{\ell} \qquad (18)
\end{aligned}$$

From (18), we see that we must choose a path of integration. If the field is curl-less ($\nabla \times \vec{E} = 0$), we can choose **any** path. Don't worry if you don't know what curl-less means, just read it as "cruel-less". In electrostatics, $\vec{E}$ is always curl-less (cruel-less), thus, we can choose any path. For simplicity, we will choose a path that is a straight line along the $x-$axis, which gives

$$V_{\text{center}} - V_{\text{surface}} = \int_{\text{center}}^{\text{surface}} \vec{E}\cdot\, dx\hat{\boldsymbol{\imath}}. \tag{19}$$

Substituting E, given by (14), into (19) yields

$$\begin{aligned}
V_{\text{center}} - V_{\text{surface}} &= \int_0^R \frac{\rho_o x^{n+1}}{(n+3)\epsilon_o}\, dx \\
V_{\text{center}} - V_{\text{surface}} &= \frac{\rho_o R^{n+2}}{(n+2)(n+3)\epsilon_o}.
\end{aligned}$$

Why did the dot product go away, and why did we replace r with x?

4. A cylinder with radius R, length ℓ, and volume charge density $\rho = \rho_o r^n$, where $\rho_o > 0$ and $n \geq 0$, is shown

in Fig. (2).

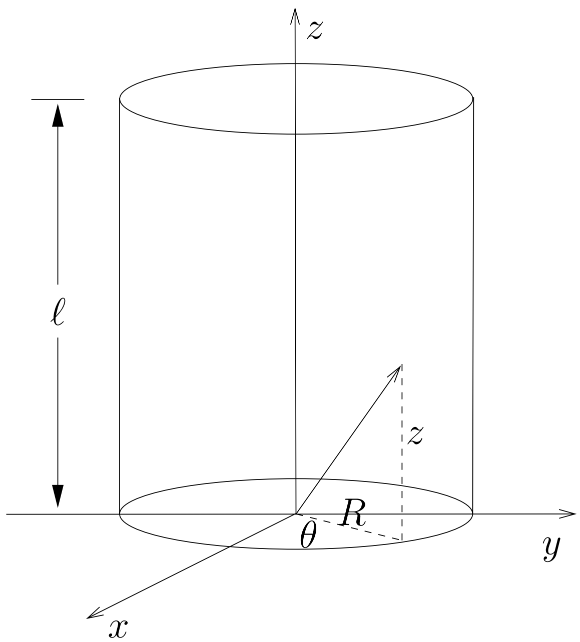

Figure 2: A cylinder with radius R, and length ℓ, where $\frac{R}{\ell} \ll 1$. The figure is not to scale.

(a) Find the electric field inside the cylinder, i.e., $r < R$.

Solution: Due to symmetry of the problem, we use Gauss's Law, given by

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\text{encl.}}}{\epsilon_o}. \tag{20}$$

We want to find the electric field inside the cylinder, thus, we choose the Gaussian surface to be a cylinder (why?) with radius $r < R$. The volume charge density distribution is cylindrically symmetrical and since $\rho_o > 0$, the electric field points radially outward in any $xy-$plane (if ρ_o was less negative, then $\vec{E}$ would point radially inward in any $xy-$plane). **We are neglecting all edge effects, by saying that the electric field is radial in any $xy-$plane. One can show that if $\frac{R}{\ell} \ll 1$, the electric field is approximately radial. To show this, you will need to expand an appropriate integrand in Taylor series.** If you are interested in working this out (perhaps for extra credit), please see me.

Guided by our qualitative description of the $\vec{E}$ field, we break up the left hand side (LHS) of (20) into three parts to yield

$$\int_{\text{wrap}} \vec{E} \cdot \mathrm{d}\vec{A} + \int_{\text{top cap}} \vec{E} \cdot \mathrm{d}\vec{A} + \int_{\text{bottom cap}} \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\text{encl.}}}{\epsilon_o}, \tag{21}$$

where "wrap" refers to the wrap-around area of the cylinder and "top/bottom caps" refers to the top/bottom caps of the cylinder. The "top/bottom caps" integrals in (21) vanish because $\vec{E}$ is perpendicular to $\mathrm{d}\vec{A}$, leaving

$$\int_{\text{wrap}} \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\text{encl.}}}{\epsilon_o}, \tag{22}$$

but $\vec{E}$ is parallel (anti-parallel if $\rho_0 < 0$) to $\mathrm{d}\vec{A}$, thus, (22) reduces to

$$\int_{\text{wrap}} E \mathrm{d}A = \frac{Q_{\text{encl.}}}{\epsilon_o}. \tag{23}$$

The electric field, E, in (23) is an unknown function, but it must be only a function of r because volume charge density only depends on r. This implies that the electric field is constant on the "wrap" surface (here we are thinking of the surface as being fixed at some r), thus we can bring E outside the integral to yield

$$\begin{aligned} E \int \mathrm{d}A &= \frac{Q_{\text{encl.}}}{\epsilon_o} \\ E\ 2\pi r \ell &= \frac{Q_{\text{encl.}}}{\epsilon_o}. \end{aligned} \tag{24}$$

Now we must compute the RHS of (24), where $Q_{\text{encl.}}$ is the amount of charge enclosed by the Gaussian surface with radius r. We recognized that we can find $Q_{\text{encl.}}$ by integrating ρ from 0 to r using a mathematical technique commonly referred to as "integration by cylindrical shells". A surface area of one cylindrical shell of radius ζ is $2\pi\zeta\ell$, thus,

$$\begin{aligned} Q_{\text{encl.}} &= \int_0^r \rho \ \mathrm{dVolume} \\ &= \int_0^r \rho \ 2\pi\zeta\ell \mathrm{d}\zeta \\ &= \int_0^r \rho_o \zeta^n \ 2\pi\zeta\ell \mathrm{d}\zeta \\ &= 2\pi\ell\rho_o \frac{r^{n+2}}{n+2} \end{aligned} \tag{25}$$

Notice that ζ was just a dummy integration variable, we could NOT use r because that is our limit of integration (annoying rule from Calculus!). Finally, substituting (25) into (24) and simplifying yields

$$E = \frac{\rho_o r^{n+1}}{(n+2)\epsilon_o}, \quad r < R. \tag{26}$$

Notice that (26) is only valid inside the cylinder, i.e., $r < R$.

(b) Find the electric field outside the cylinder, i.e., $r > R$.

Solution: We can find the electric field outside the cylinder, i.e., $r > R$, as in part (a) but we have to be more careful when we compute $Q_{\text{encl.}}$ since our Gaussian surface lies outside the physical cylinder, and $\rho = 0$ for $r > R$. In other words,

$$\begin{aligned} Q_{\text{encl.}} &= \int_0^r \rho \, \text{dVolume} \\ &= \int_0^R \rho \; 2\pi\zeta\ell \text{d}\zeta + \int_R^r \rho \; 2\pi\zeta\ell \text{d}\zeta \\ &= \int_0^R \rho_o \zeta^n \; 2\pi\zeta\ell \text{d}\zeta + \int_R^r 0 \; 2\pi\zeta\ell \text{d}\zeta \\ &= 2\pi\ell\rho_o \frac{R^{n+2}}{n+2}. \end{aligned} \tag{27}$$

Finally, substituting (27) into (24) and simplifying yields

$$E = \frac{\rho_o R^{n+2}}{(n+2) r \epsilon_o}, \quad r > R. \tag{28}$$

Notice that (28) is only valid outside the cylinder, i.e., $r > R$.

(c) Find the electric potential difference between the center and the surface of the cylinder, i.e., $V_{\text{center}} - V_{\text{surface}}$.

Solution: The potential difference is given by (see part (c), equation (18))

$$V_{\text{center}} - V_{\text{surface}} = \int_{\text{center}}^{\text{surface}} \vec{E} \cdot \, \text{d}\vec{\ell} \tag{29}$$

From (29), we see that we must choose a path of integration. If the field is curl-less ($\nabla \times \vec{E} = 0$), we can choose **any** path. Don't worry if you don't know what curl-less means, just read it as "cruel-less". In electrostatics, $\vec{E}$ is always curl-less (cruel-less), thus, we can choose any path. For simplicity, we will choose a path that is a straight line along the $x-$axis, giving

$$V_{\text{center}} - V_{\text{surface}} = \int_{\text{center}}^{\text{surface}} \vec{E} \cdot \, \text{d}x\hat{\boldsymbol{\imath}}. \tag{30}$$

Substituting E, given by (26), into (30) yields

$$\begin{aligned} V_{\text{center}} - V_{\text{surface}} &= \int_0^R \frac{\rho_o x^{n+1}}{(n+2)\epsilon_o} \, \text{d}x \\ V_{\text{center}} - V_{\text{surface}} &= \frac{\rho_o R^{n+2}}{(n+2)^2 \epsilon_o}. \end{aligned}$$

Why did the dot product go away, and why did we replace r with x?