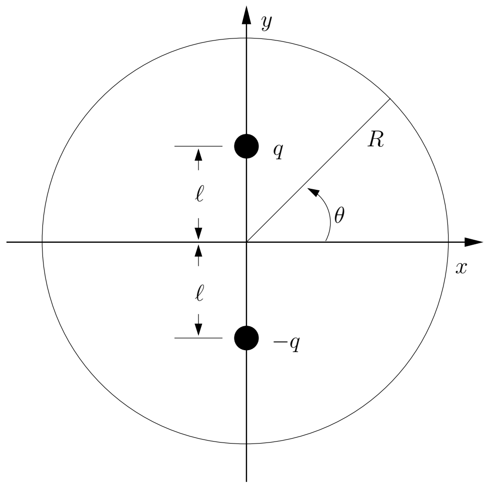

Figure 1: For the above electric dipole, $p = 2\ell q$.

1. Dipole in a 2-D world because the 3-D world is too damn hard!
   (a) Find the electric field anywhere in the xy-plane (see Fig. 1).

Solution: The electric field due to N point particles is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\sum_{i=1}^{N} q_i \frac{\vec{r}_i}{{r_i}^3}, \quad \text{where } \vec{r} = \vec{r}_f - \vec{r}_{s_i}. \tag{1}$$

$\vec{r}_f$ is called the **field point** and $\vec{r}_{s_i}$ is the i^{th} **source point**.
We will call the positive charge the first charge and the negative charge the second charge. Thus,

$$\vec{r}_f = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \qquad \vec{r}_{s_1} = \ell\hat{\boldsymbol{\jmath}} \qquad \vec{r}_{s_2} = -\ell\hat{\boldsymbol{\jmath}},$$

Computing all quantities needed for (1) yields

$$\begin{aligned} \vec{r}_1 &= \vec{r}_f - \vec{r}_{s_1} & \vec{r}_2 &= \vec{r}_f - \vec{r}_{s_2} \\ &= x\hat{\boldsymbol{\imath}} + (y-\ell)\hat{\boldsymbol{\jmath}} & &= x\hat{\boldsymbol{\imath}} + (y+\ell)\hat{\boldsymbol{\jmath}} \quad (2) \\ r_1 &= ||\vec{r}_f - \vec{r}_{s_1}|| & r_2 &= ||\vec{r}_f - \vec{r}_{s_2}|| \\ &= \sqrt{x^2 + (y-\ell)^2} & &= \sqrt{x^2 + (y+\ell)^2}. \quad (3) \end{aligned}$$

Finally, substituting (2) and (3) into (1) yields

$$\vec{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{x\hat{\boldsymbol{\imath}} + (y-\ell)\hat{\boldsymbol{\jmath}}}{[x^2 + (y-\ell)^2]^{3/2}} - \frac{x\hat{\boldsymbol{\imath}} + (y+\ell)\hat{\boldsymbol{\jmath}}}{[x^2 + (y+\ell)^2]^{3/2}}\right] \tag{4}$$

(b) Evaluate the electric field found in part (a) on a circle with radius R (see Fig. 1).

Solution: To find the electric field on the circle, we set the field point on the circle, i.e., $x = R\cos\theta$ and $y = R\sin\theta$ in (4), which yields

$$\vec{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{R\cos\theta\hat{\boldsymbol{\imath}} + (R\sin\theta - \ell)\hat{\boldsymbol{\jmath}}}{[R^2 - 2\ell R\sin\theta + \ell^2]^{3/2}} - \frac{R\cos\theta\hat{\boldsymbol{\imath}} + (R\sin\theta + \ell)\hat{\boldsymbol{\jmath}}}{[R^2 + 2\ell R\sin\theta + \ell^2]^{3/2}}\right] \tag{5}$$

(c) Find an approximate expression for the **x-component** of the electric field on the circle if $\ell/R \ll 1$ (see Fig. 1). Hint: Expand the denominator in Taylor series, $(1+\epsilon)^n = 1 + n\epsilon + \cdots$, if $|\epsilon| < 1$. You can drop any terms containing square or higher powers of $\frac{\ell}{R}$ because if $\frac{\ell}{R}$ is small, then $\left(\frac{\ell}{R}\right)^2$ is super-small.

Solution: From (5), we see that the x-component of the electric field on the circle is given by

$$E_x = \frac{qR\cos\theta}{4\pi\epsilon_o}\left[\frac{1}{[R^2 - 2\ell R\sin\theta + \ell^2]^{3/2}} - \frac{1}{[R^2 + 2\ell R\sin\theta + \ell^2]^{3/2}}\right]. \tag{6}$$

We rewrite the denominators as follows:

$$\begin{aligned}
\frac{1}{[R^2 \mp 2\ell R\sin\theta + \ell^2]^{3/2}} &= \left[R^2 \mp 2\ell R\sin\theta + \ell^2\right]^{-3/2} \\
&= \left[R^2\left(1 \mp \frac{2\ell\sin\theta}{R} + \left(\frac{\ell}{R}\right)^2\right)\right]^{-3/2} \\
&= R^{-3}\left[1 \mp \frac{2\ell\sin\theta}{R} + \left(\frac{\ell}{R}\right)^2\right]^{-3/2} \\
&= R^{-3}\left[1+\epsilon\right]^{-3/2}, \text{ where } \epsilon = \mp\frac{2\ell\sin\theta}{R} + \left(\frac{\ell}{R}\right)^2 \\
&= R^{-3}\left[1 - \frac{3}{2}\epsilon + \cdots\right] \\
&= R^{-3}\left[1 \pm \frac{3\ell\sin\theta}{R} - \frac{3}{2}\left(\frac{\ell}{R}\right)^2 + \cdots\right] \\
&\approx R^{-3}\left[1 \pm \frac{3\ell\sin\theta}{R}\right].
\end{aligned}$$

Finally, substituting the above result into (6) and simplifying yields

$$E_x = \frac{3p}{4\pi\epsilon_o R^3}\sin(\theta)\cos(\theta), \quad \text{where} \quad p = 2\ell q.$$

(d) Find the total charge enclosed by the circle. Find the unit-normal to the circle.

Solution: The total charge enclosed by the circle is given by

$$\begin{aligned} q_{\text{enclosed}} &= q_1 + q_2 \\ &= q + (-q) \\ &= 0 \end{aligned}$$

We will find the radially-outward unit-normal for the upper-half of the circle (you should do both!). The equation for the circle (upper-half) with radius R is given by

$$R = \sqrt{x^2 + y^2}.$$

The non-unit normal is given by (review Calculus!)

$$\begin{aligned} \hat{n} &= \frac{\partial R}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial R}{\partial y}\hat{\boldsymbol{\jmath}} \\ &= \frac{1}{\sqrt{x^2+y^2}}\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}\right). \end{aligned} \tag{7}$$

Substituting $x = R\cos\theta$ and $y = R\sin\theta$ into (7) yields

$$\vec{n} = \cos(\theta)\,\hat{\boldsymbol{\imath}} + \sin(\theta)\,\hat{\boldsymbol{\jmath}}. \tag{8}$$

It is trivial to confirm that $\vec{n}$ given in (8) is actually $\hat{n}$ (we got lucky!).

2. Hard integrals made easy!

   (a) Set-up an integral expression for the electric field at a field point, $(x_o, 0)$, due to a ring of charge with linear charge density $\lambda = \lambda_o \sin\theta$, where λ_o is a **positive constant** (see Fig. 2).

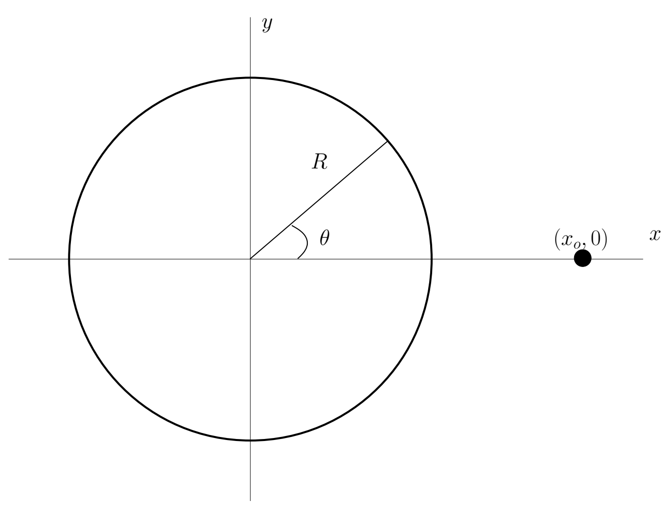

Figure 2: The circle has a linear charge density given by $\lambda = \lambda_o \sin\theta$, where λ_o is a **positive constant**.

Solution: The electric field is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_o} \int \frac{\vec{r}}{r^3}\, \mathrm{d}q, \tag{9}$$

where $\vec{r} = \vec{r}_f - \vec{r}_s$. From Fig. 2, we have

$$\begin{aligned} \vec{r}_f &= x_o\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} \\ \vec{r}_s &= R\cos(\theta)\hat{\boldsymbol{\imath}} + R\sin(\theta)\hat{\boldsymbol{\jmath}}. \end{aligned}$$

Thus,

$$\vec{r} = (x_o - R\cos\theta)\,\hat{\boldsymbol{\imath}} - R\sin\theta\hat{\boldsymbol{\jmath}} \tag{10}$$

$$r = \sqrt{R^2 + x_o^2 - 2x_oR\cos\theta}. \tag{11}$$

dq is given by

$$\mathrm{d}q = \lambda \mathrm{d}r_s, \tag{12}$$

where dr_s is the differential element of the source variable (we are summing up the source points, NOT the field points, hence, we use r_s NOT r_f). Also, note that dq is a scalar, thus, the right hand side of (12) must also be a scalar. One way to find dr_s is as follows:

$$\begin{aligned} \vec{r}_s &= R\cos(\theta)\hat{\boldsymbol{\imath}} + R\sin(\theta)\hat{\boldsymbol{\jmath}} \\ \frac{\mathrm{d}\vec{r}_s}{\mathrm{d}\theta} &= -R\sin(\theta)\hat{\boldsymbol{\imath}} + R\cos(\theta)\hat{\boldsymbol{\jmath}} \\ \frac{\mathrm{d}r_s}{\mathrm{d}\theta} &= \left|\left|\frac{\mathrm{d}\vec{r}_s}{\mathrm{d}\theta}\right|\right| \\ \frac{\mathrm{d}r_s}{\mathrm{d}\theta} &= \sqrt{R^2\left(\cos^2\theta + \sin^2\theta\right)} \\ \frac{\mathrm{d}r_s}{\mathrm{d}\theta} &= R \\ \mathrm{d}r_s &= R\mathrm{d}\theta. \end{aligned} \tag{13}$$

Finally, substituting (10), (11), and (13) into (9) yields

$$\vec{E} = \frac{R}{4\pi\epsilon_o}\left[\int_0^{2\pi} \frac{\lambda\,(x_o - R\cos\theta)}{\left(R^2 + x_o^2 - 2x_oR\cos\theta\right)^{3/2}}\mathrm{d}\theta\,\hat{\boldsymbol{\imath}} - \int_0^{2\pi} \frac{\lambda R\sin\theta}{\left(R^2 + x_o^2 - 2x_oR\cos\theta\right)^{3/2}}\mathrm{d}\theta\,\hat{\boldsymbol{\jmath}}\right], \tag{14}$$

where $\lambda = \lambda_o \sin\theta$.

(b) Using only a symmetry argument, find the direction of the electric field.

Solution:

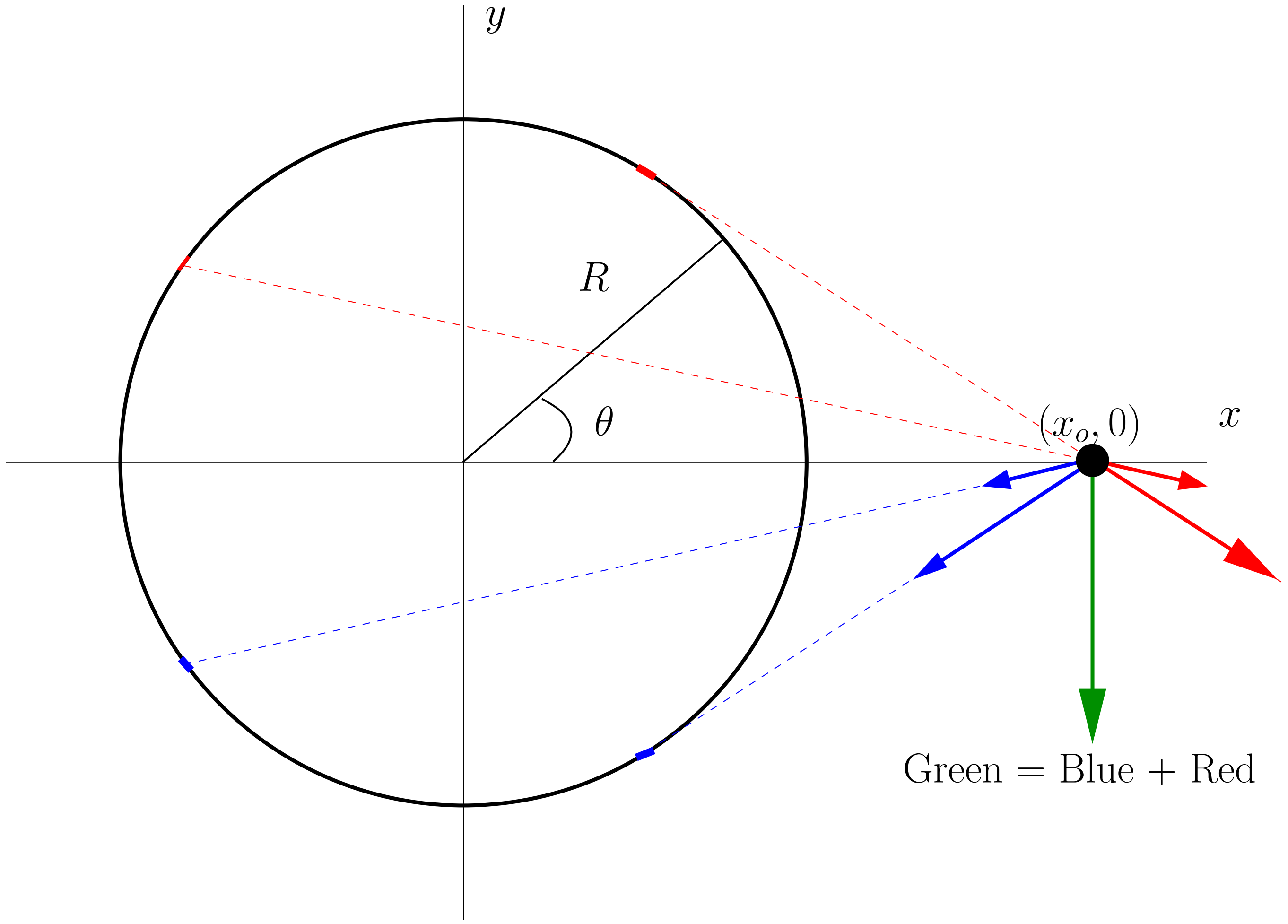

Figure 3: We know that the electric field points radially outward from a positive charge (shown in red) and radially inward from a negative charge (shown in blue). The linear charge density is given by $\lambda = \lambda_o \sin\theta$, thus, the upper half of the circle is positively charged and the lower half of the circle is negatively charged. By drawing a few "representative" charges on the circle, we readily see that the electric field has ONLY a $y-$component. Moreover, we see that the electric field points down, i.e., in the negative y direction. It is interesting to note that we have effectively calculated the $x-$component of the integral given by (14) just by drawing a picture!

3. Given a curve $y = f(x)$ where $x_1 \leq x \leq x_2$ in the $xy-$plane, find the electric field at some field point, (a, b). Assume that the curve has a linear charge density given by $\lambda = g(x)$.

Solution: The electric field is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_o} \int \frac{\vec{r}}{r^3}\, \mathrm{d}q, \tag{15}$$

where $\vec{r} = \vec{r}_f - \vec{r}_s$. First we find $\vec{r}_f$ and $\vec{r}_s$ as follows:

$$\begin{aligned}\vec{r}_f &= a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} \\ \vec{r}_s &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \\ &= x\hat{\boldsymbol{\imath}} + f(x)\hat{\boldsymbol{\jmath}} \quad \text{recall that } y = f(x)\end{aligned}$$

Thus,

$$\vec{r} = [a - x]\,\hat{\boldsymbol{\imath}} + [b - f(x)]\,\hat{\boldsymbol{\jmath}} \tag{16}$$

$$r = \sqrt{[a - x]^2 + [b - f(x)]^2}. \tag{17}$$

$\mathrm{d}q$ is given by $\mathrm{d}q = \lambda \mathrm{d}r_s$ and can be found in the usual manner,

$$\begin{aligned}\vec{r}_s &= x\hat{\boldsymbol{\imath}} + f(x)\hat{\boldsymbol{\jmath}} \\ \frac{\mathrm{d}\vec{r}_s}{\mathrm{d}x} &= 1\hat{\boldsymbol{\imath}} + f'(x)\hat{\boldsymbol{\jmath}} \\ \frac{\mathrm{d}r_s}{\mathrm{d}x} &= \left\|\frac{\mathrm{d}\vec{r}_s}{\mathrm{d}x}\right\| \\ &= \sqrt{1 + [f'(x)]^2} \\ \mathrm{d}r_s &= \sqrt{1 + [f'(x)]^2}\mathrm{d}x. \end{aligned} \tag{18}$$

Note that (18) is just an arc length formula from Calculus. Finally, substituting (16), (17) and (18) into (15) yields

$$\begin{aligned}\vec{E} = &\frac{1}{4\pi\epsilon_o} \int_{x_1}^{x_2} \frac{\lambda(a - x)}{\left[(a - x)^2 + (b - f(x))^2\right]^{3/2}} \sqrt{1 + [f'(x)]^2}\, \mathrm{d}x\, \hat{\boldsymbol{\imath}} \\ &+ \frac{1}{4\pi\epsilon_o} \int_{x_1}^{x_2} \frac{\lambda(b - f(x))}{\left[(a - x)^2 + (b - f(x))^2\right]^{3/2}} \sqrt{1 + [f'(x)]^2}\, \mathrm{d}x\, \hat{\boldsymbol{\jmath}},\end{aligned} \tag{19}$$

where $\lambda = g(x)$.