Name: ____________________

CWID: ____________________

Question	Points	Score
1	10	
2	10	
Total:	20	

Calculators Not Allowed
No Work = No Credit
Write Legibly

1. [10 points] Using only a symmetry argument, find the direction of the electric field at a field point, $(0, y_o)$, due to a ring of charge with linear charge density $\lambda = -\lambda_o \cos\theta$, where λ_o is a **positive constant** (see Fig. 1). **To receive full credit you must explain your symmetry argument clearly.**

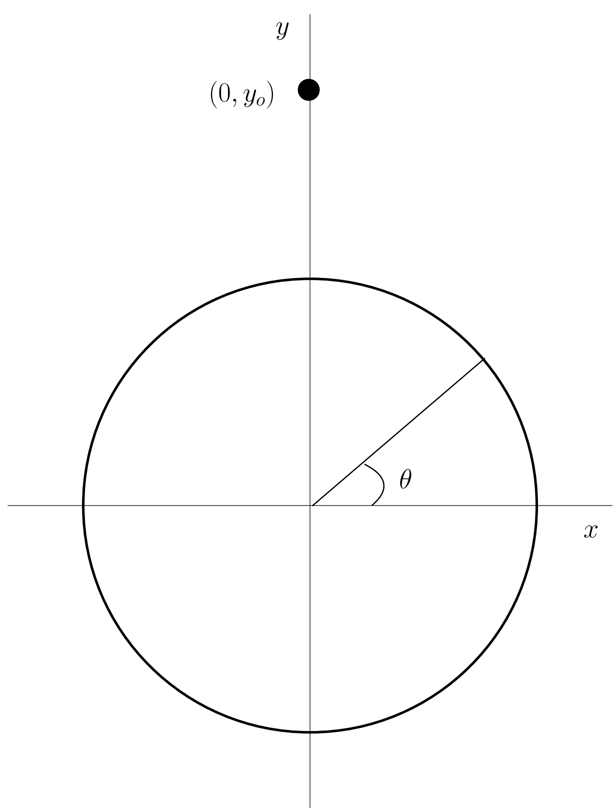

Figure 1: The circle has a linear charge density given by $\lambda = -\lambda_o \cos\theta$, where λ_o is a **positive constant**.

Solution:

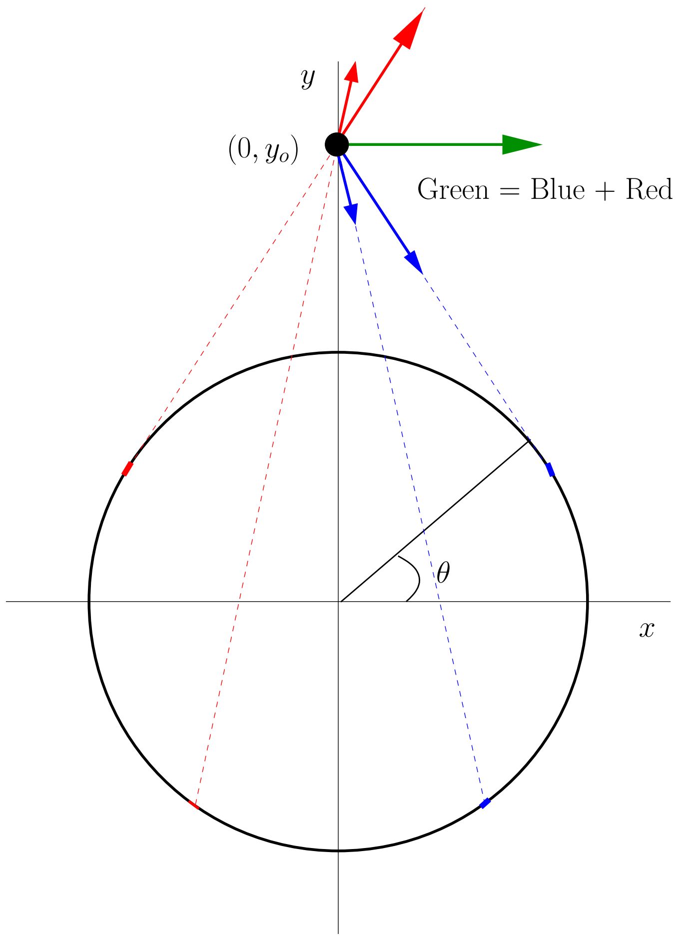

Figure 2: We know that the electric field points radially outward from a positive charge (shown in red) and radially inward from a negative charge (shown in blue). The linear charge density is given by $\lambda = -\lambda_o \cos\theta$, thus, the half of the circle on the left hand side is positively charged and the half of the circle on the right hand side is negatively charged. By drawing a few "representative" charges on the circle, we readily see that the electric field has ONLY an $x-$component. Moreover, we see that the electric field points to the right, i.e., in the positive x direction.

Table 1: For Grader Use Only: Rough grading criteria are given below.

charge distribution	3 pt
student drew some vectors	2 pt
clear explanation	5 pt

2. Dipole in a 2-D world because the 3-D world is too damn hard!

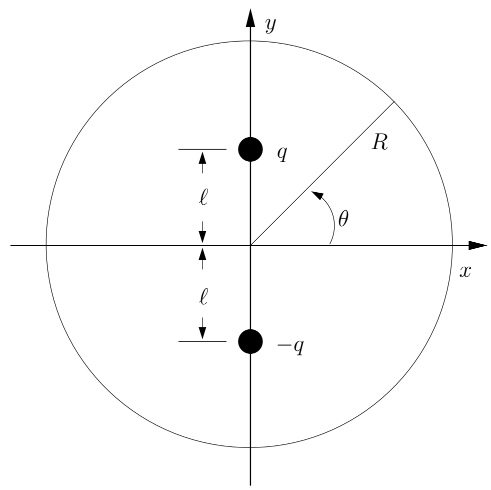

Figure 3: For the above electric dipole, $p = 2\ell q$.

(a) 6 points Find the electric field on a circle with radius R due to a dipole (see Fig. 3).

Solution: The electric field due to N point particles is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\sum_{i=1}^{N} q_i \frac{\vec{r}_i}{{r_i}^3}, \quad \text{where } \vec{r} = \vec{r}_f - \vec{r}_{s_i}. \tag{1}$$

$\vec{r}_f$ is called the **field point** and $\vec{r}_{s_i}$ is the i^{th} **source point**.
We will call the positive charge the first charge and the negative charge the second charge. Thus,

$$\vec{r}_f = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \qquad \vec{r}_{s_1} = \ell\hat{\boldsymbol{\jmath}} \qquad \vec{r}_{s_2} = -\ell\hat{\boldsymbol{\jmath}},$$

Computing all quantities needed for (1) yields

$$\begin{aligned} \vec{r}_1 &= \vec{r}_f - \vec{r}_{s_1} & \vec{r}_2 &= \vec{r}_f - \vec{r}_{s_2} \\ &= x\hat{\boldsymbol{\imath}} + (y-\ell)\hat{\boldsymbol{\jmath}} & &= x\hat{\boldsymbol{\imath}} + (y+\ell)\hat{\boldsymbol{\jmath}} \qquad (2) \\ r_1 &= \|\vec{r}_f - \vec{r}_{s_1}\| & r_2 &= \|\vec{r}_f - \vec{r}_{s_2}\| \\ &= \sqrt{x^2 + (y-\ell)^2} & &= \sqrt{x^2 + (y+\ell)^2}. \qquad (3) \end{aligned}$$

Finally, substituting (2) and (3) into (1) yields

$$\vec{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{x\hat{\boldsymbol{\imath}} + (y-\ell)\hat{\boldsymbol{\jmath}}}{[x^2 + (y-\ell)^2]^{3/2}} - \frac{x\hat{\boldsymbol{\imath}} + (y+\ell)\hat{\boldsymbol{\jmath}}}{[x^2 + (y+\ell)^2]^{3/2}}\right] \tag{4}$$

To find the electric field on the circle, we set the field point on the circle, i.e., $x = R\cos\theta$ and $y = R\sin\theta$ in (4), which yields

$$\vec{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{R\cos\theta\hat{\boldsymbol{\imath}} + (R\sin\theta - \ell)\hat{\boldsymbol{\jmath}}}{[R^2 - 2\ell R\sin\theta + \ell^2]^{3/2}} - \frac{R\cos\theta\hat{\boldsymbol{\imath}} + (R\sin\theta + \ell)\hat{\boldsymbol{\jmath}}}{[R^2 + 2\ell R\sin\theta + \ell^2]^{3/2}}\right] \tag{5}$$

Table 2: For Grader Use Only: Rough grading criteria are given below.

$\vec{r}_f$	1 pt
$\vec{r}_s$	2 pt
$\vec{r}$	1 pt
other computations	2 pt

(b) 4 points Find an approximate expression for the **y-component** of the electric field on the circle if $\ell/R \ll 1$ (see Fig. 3). Hint: Expand the denominator in Taylor series, $(1+\epsilon)^n = 1 + n\epsilon + \cdots$, if $|\epsilon| < 1$. You can drop any terms containing square or higher powers of $\frac{\ell}{R}$ because if $\frac{\ell}{R}$ is small, then $\left(\frac{\ell}{R}\right)^2$ is super-small. You may find the following formulas useful: $(a \mp c)(1 \pm b) = a \pm ab \mp c - bc$ and $(a + ab - c - bc) - (a - ab + c - bc) = 2(ab - c)$. It is interesting to note that $\vec{E}\cdot\hat{n} = \frac{2p\sin\theta}{4\pi\epsilon_o R^3}$.

Solution: From (5), we see that the y-component of the electric field on the circle is given by

$$E_y = \frac{q}{4\pi\epsilon_o}\left[\frac{R\sin\theta - \ell}{[R^2 - 2\ell R\sin\theta + \ell^2]^{3/2}} - \frac{R\sin\theta + \ell}{[R^2 + 2\ell R\sin\theta + \ell^2]^{3/2}}\right]. \tag{6}$$

We rewrite the denominators as follows:

$$\begin{aligned}
\frac{1}{[R^2 \mp 2\ell R\sin\theta + \ell^2]^{3/2}} &= \left[R^2 \mp 2\ell R\sin\theta + \ell^2\right]^{-3/2} \\
&= \left[R^2\left(1 \mp \frac{2\ell\sin\theta}{R} + \left(\frac{\ell}{R}\right)^2\right)\right]^{-3/2} \\
&= R^{-3}\left[1 \mp \frac{2\ell\sin\theta}{R} + \left(\frac{\ell}{R}\right)^2\right]^{-3/2} \\
&= R^{-3}\left[1+\epsilon\right]^{-3/2}, \text{ where } \epsilon = \mp\frac{2\ell\sin\theta}{R} + \left(\frac{\ell}{R}\right)^2 \\
&= R^{-3}\left[1 - \frac{3}{2}\epsilon + \cdots\right] \\
&= R^{-3}\left[1 \pm \frac{3\ell\sin\theta}{R} - \frac{3}{2}\left(\frac{\ell}{R}\right)^2 + \cdots\right] \\
&\approx R^{-3}\left[1 \pm \frac{3\ell\sin\theta}{R}\right].
\end{aligned}$$

Finally, substituting the above result into (6) and simplifying yields

$$E_y = \frac{p}{4\pi\epsilon_o R^3}\left[3\sin^2(\theta) - 1\right], \quad \text{where} \quad p = 2\ell q. \tag{7}$$

To obtain (7), we have used the hint-formulas given in the statement of the problem with $a = R\sin\theta,\ c = \ell,\ b = \frac{3\ell\sin\theta}{R}$.

Table 3: For Grader Use Only: Rough grading criteria are given below.

Taylor expansion	3 pt
other algebra	1 pt