1. (15 points) Evaluate

$$\int_0^2 \int_1^3 \int_0^{1-z^2} 5z\mathrm{e}^{3y}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y.$$

Solution:

$$\int_0^2 \int_1^3 \int_0^{1-z^2} 5z\mathrm{e}^{3y}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y = \int_0^2 \int_1^3 5zx\mathrm{e}^{3y}\Big|_{x=0}^{x=1-z^2}\,\mathrm{d}z\,\mathrm{d}y = \int_0^2 \int_1^3 5\mathrm{e}^{3y}\left(z - z^3\right)\mathrm{d}z\,\mathrm{d}y$$

$$= \int_0^2 5\mathrm{e}^{3y}\left(\frac{z^2}{2} - \frac{z^4}{4}\right)\Bigg|_{z=1}^{z=3}\mathrm{d}y = -80\int_0^2 \mathrm{e}^{3y}\,\mathrm{d}y$$

$$= -\frac{80}{3}\mathrm{e}^{3y}\Bigg|_{y=0}^{y=2} = \boxed{-\frac{80}{3}\left(\mathrm{e}^6 - 1\right)}$$

2. (10 points) Evaluate

$$\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \left(x^2z + y^2z + z^3\right)\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

Solution:

$$\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \left(x^2z + y^2z + z^3\right)\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} \left(\frac{x^2z^2}{2} + \frac{y^2z^2}{2} + \frac{z^4}{4}\right)\Bigg|_{z=-\sqrt{a^2-x^2-y^2}}^{z=\sqrt{a^2-x^2-y^2}}\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} 0\,\mathrm{d}x\,\mathrm{d}y = \boxed{0}$$

You could have seen that the integral is going to vanish because the integrand in an odd function in z and you are integrating on a symmetrical interval.

3. (20 points) Find the surface area of the part of the paraboloid $y = 3x^2 + 3z^2$ that lies inside $x^2 + z^2 = 2$.

Solution:

$$\begin{aligned}
\text{Surface Area} &= \iint_D \sqrt{(y_x)^2 + (y_z)^2 + 1}\, \mathrm{d}x\, \mathrm{d}z, \quad \text{where } D \text{ is a disk of radius } \sqrt{2} \\
&= \iint_D \sqrt{(6x)^2 + (6z)^2 + 1}\, \mathrm{d}x\, \mathrm{d}z \\
&= \iint_D \sqrt{36\left(x^2 + z^2\right) + 1}\, \mathrm{d}x\, \mathrm{d}z \qquad (1)
\end{aligned}$$

Substituting $x = r\cos\theta$, $z = r\sin\theta$ and $\mathrm{d}x\, \mathrm{d}z = r\, \mathrm{d}r\, \mathrm{d}\theta$ into (1) yields

$$\begin{aligned}
\text{Surface Area} &= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{36r^2 + 1}\, r\, \mathrm{d}r\, \mathrm{d}\theta \\
&= \frac{1}{72} \int_0^{2\pi} \int_1^{73} \sqrt{u}\, \mathrm{d}u\, \mathrm{d}\theta, \quad \text{where } u = 36r^2 + 1 \text{ and } \mathrm{d}u = 72\, r\, \mathrm{d}r \\
&= \boxed{\frac{\pi}{54}\left(73\sqrt{73} - 1\right).}
\end{aligned}$$

4. (10 points) Evaluate

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r},$$

where $\vec{F}(x, y) = 2xy\hat{\boldsymbol{\imath}} + x^2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ and the curve C is given by $\vec{r}(t) = t^2\hat{\boldsymbol{\imath}} + t^3\hat{\boldsymbol{\jmath}} + t\hat{\boldsymbol{k}}$, $0 \le t \le 1$.

Solution:

$$\vec{r} = t^2\hat{\boldsymbol{\imath}} + t^3\hat{\boldsymbol{\jmath}} + t\hat{\boldsymbol{k}} \qquad \vec{F}\left(\vec{r}(t)\right) = 2t^5\hat{\boldsymbol{\imath}} + t^4\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$$

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = 2t\hat{\boldsymbol{\imath}} + 3t^2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}} \qquad \vec{F} \cdot \mathrm{d}\vec{r} = \left(2t^5\hat{\boldsymbol{\imath}} + t^4\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\right) \cdot \left(2t\hat{\boldsymbol{\imath}} + 3t^2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\right)$$

$$\mathrm{d}\vec{r} = \left(2t\hat{\boldsymbol{\imath}} + 3t^2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\right)\mathrm{d}t \qquad \int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^1 \left(7t^6 + 1\right)\mathrm{d}t = \boxed{2}$$

5. (20 points) Use spherical coordinates to find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$ and above the xy-plane and below the cone $z=\sqrt{\frac{x^2}{3}+\frac{y^2}{3}}$.

Solution: Figure 1 shows the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$ and above the xy-plane and below the cone $z=\sqrt{\frac{x^2}{3}+\frac{y^2}{3}}$.

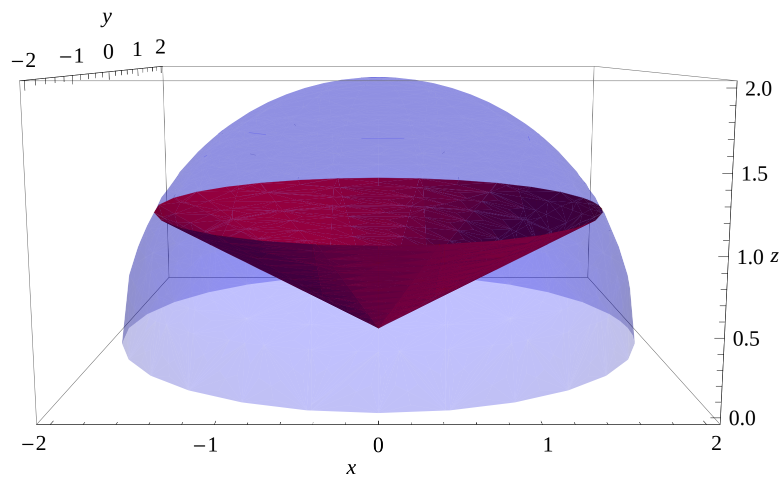

Figure 1: The half-sphere is shown in light blue color and the cone is shown in red color.

Substituting $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$ and $z=\rho\cos\phi$ into $x^2+y^2+z^2=4$ and $z=\sqrt{\frac{x^2}{3}+\frac{y^2}{3}}$ yields

$$\begin{aligned}\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi&=4\\ \rho^2\sin^2\phi+\rho^2\cos^2\phi&=4\\ \rho&=2\text{ (sphere)}\end{aligned}\qquad\qquad \begin{aligned}\rho\cos\phi&=\sqrt{\frac{\rho^2\sin^2\phi}{3}}\\ \tan\phi&=\sqrt{3}\\ \phi&=\frac{\pi}{3}\text{ (cone).}\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^2\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta&=\frac{8}{3}\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\\ &=\frac{8}{3}\int_0^{2\pi}\frac{1}{2}\,\mathrm{d}\theta\\ &=\boxed{\frac{8\pi}{3}}\end{aligned}$$

6. (10 points) A particle starts at the point $(-3,0)$ moves along the x-axis to $(3,0)$ and then along the semicircle $y=\sqrt{9-x^2}$ to the starting point. Use Green's Theorem to find the work done on the particle by the force field

$$\vec{F}(x,y) = 24x\,\hat{\boldsymbol{\imath}} + \left(8x^3 + 24xy^2\right)\hat{\boldsymbol{\jmath}}.$$

Solution: We apply Green's Theorem,

$$\oint \vec{F}\cdot \mathrm{d}\vec{r} = \iint \left(\nabla \times \vec{F}\right)\cdot \hat{\boldsymbol{k}}\,\mathrm{d}A,$$

to the region shown in Figure 2 to obtain

$$\left(\nabla \times \vec{F}\right)\cdot \hat{\boldsymbol{k}} = \frac{\partial}{\partial x}\left(8x^3 + 24xy^2\right) - \frac{\partial}{\partial y}\left(24x\right) = 24x^2 + 24y^2$$

$$\oint \vec{F}\cdot \mathrm{d}\vec{r} = \iint \left(24x^2 + 24y^2\right)\mathrm{d}A = \int_0^{\pi}\int_0^{3} 24r^2\, r\,\mathrm{d}r\,\mathrm{d}\theta = \boxed{486\pi.}$$

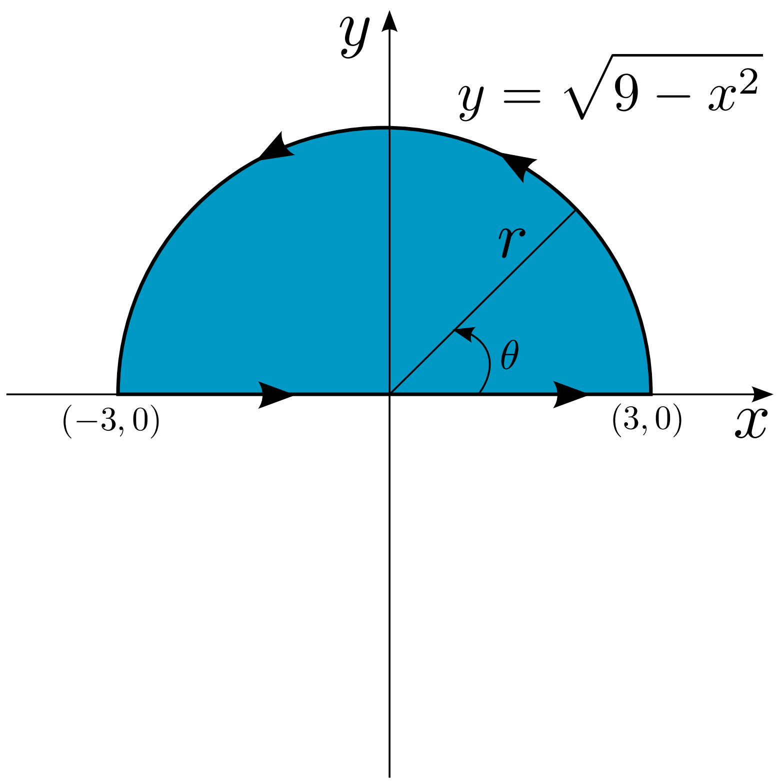

Figure 2: The region used in Green's Theorem is shown in blue color.

7. (5 points) Find the Jacobian of the transformation, $x = 5\alpha\sin\beta$, $y = 4\alpha\cos\beta$.

Solution: The Jacobian is given by

$$J = \left\| \begin{matrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{matrix} \right\| = \left\| \begin{matrix} 5\sin\beta & 5\alpha\cos\beta \\ 4\cos\beta & -4\alpha\sin\beta \end{matrix} \right\| = -20\alpha\sin^2\beta - 20\alpha\cos^2\beta = \boxed{-20\alpha.}$$

8. (10 points) If a force $\vec{F}$ is given by

$$\vec{F} = \left(x^2 + y^2 + z^2\right)^n \langle x, y, z \rangle,$$

where $n \geq 1$.

(a) Find the curl $\vec{F}$.

Solution: Let $s = x^2 + y^2 + z^2$ then

$$\begin{aligned} \nabla \times \vec{F} &= \hat{\boldsymbol{\imath}} \left[\frac{\partial}{\partial y}(zs^n) - \frac{\partial}{\partial z}(ys^n) \right] - \hat{\boldsymbol{\jmath}} \left[\frac{\partial}{\partial x}(zs^n) - \frac{\partial}{\partial z}(xs^n) \right] + \hat{\boldsymbol{k}} \left[\frac{\partial}{\partial x}(ys^n) - \frac{\partial}{\partial y}(xs^n) \right] \\ &= \hat{\boldsymbol{\imath}} \left[nzs^{n-1}s_y - nys^{n-1}s_z \right] - \hat{\boldsymbol{\jmath}} \left[nzs^{n-1}s_x - nxs^{n-1}s_z \right] + \hat{\boldsymbol{k}} \left[nys^{n-1}s_x - nxs^{n-1}s_y \right]. \end{aligned} \tag{2}$$

Substituting $s_x = 2x$, $s_y = 2y$ and $s_z = 2z$ into (2) yields

$$\boxed{\nabla \times \vec{F} = \vec{0}.}$$

(b) Find the potential function, f.

Solution: A potential function exists because $\nabla \times \vec{F} = \vec{0}$. Thus, we can write $\vec{F} = \nabla f(x, y, z)$ which yields

$$f_x(x, y, z) = x\left(x^2 + y^2 + z^2\right)^n \tag{3}$$
$$f_y(x, y, z) = y\left(x^2 + y^2 + z^2\right)^n \tag{4}$$
$$f_z(x, y, z) = z\left(x^2 + y^2 + z^2\right)^n \tag{5}$$

Integrating (3) w.r.t x with $u = x^2 + y^2 + z^2$ and $\mathrm{d}u = 2x\mathrm{d}x$ yields

$$f(x, y, z) = \frac{\left(x^2 + y^2 + z^2\right)^{n+1}}{2(n+1)} + g(y, z), \tag{6}$$

where $g(y, z)$ is an arbitrary function of y and z. Differentiating (6) w.r.t y yields

$$f_y(x, y, z) = y\left(x^2 + y^2 + z^2\right)^n + g_y(y, z). \tag{7}$$

Equating (7) with (4) yields $g_y(y, z) = 0$. Integrating $g_y(y, z) = 0$ w.r.t y yields $g(y, z) = h(z)$, where $h(z)$ is an arbitrary function of z. Substituting $g(y, z) = h(z)$ into (6) and differentiating w.r.t z yields

$$f_z(x, y, z) = z\left(x^2 + y^2 + z^2\right)^n + h'(z). \tag{8}$$

Finally, equating (8) with (5) yields $h'(z) = 0$, thus, $h(z) = K$, where K is a constant of integration. Therefore,

$$\boxed{f(x, y, z) = \frac{\left(x^2 + y^2 + z^2\right)^{n+1}}{2n+2} + K, \quad \text{where } K \text{ is a constant of integration.}}$$

(c) In part (b) let $n = 0$ and use the Fundamental Theorem for Line Integrals to find the work done by the force $\vec{F}$ on a particle when it moves along the line segment from $(0, 4, 0)$ to $(0, 0, 2)$.

Solution: For $n = 0$ case, we still have $\nabla \times \vec{F} = \vec{0}$, thus, the field is still conservative. Therefore,

$$W = \int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \nabla f(x, y, z) \cdot \mathrm{d}\vec{r} = f(0, 0, 2) - f(0, 4, 0) = \left(\frac{2^2}{2} + K\right) - \left(\frac{4^2}{2} + K\right) = \boxed{-6}.$$