1. (6 points) For $f(x,y)=\sqrt{(x+3)^2+(y-1)^2}$, find

   (a) the domain

Solution:

$$\boxed{\{(x,y)|x\in\mathbb{R},y\in\mathbb{R}\}}$$

   (b) the range

Solution:

$$\boxed{\{z=f(x,y)|z\geq 0\}}$$

2. (12 points) Given $R(u,v,w)=\ln\left(u^2+v^2+w^2\right)$ and $u=xe^{xy}$, $v=\frac{e^{xy}}{y}$ and $w=\frac{1}{1+\frac{x^2}{y^2}}$. Find $\frac{\partial R}{\partial y}$ when $x=0$ and $y=1$.

Solution: Partial derivative of R w.r.t y is given by

$$\frac{\partial R}{\partial y}=\frac{\partial R}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial R}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial R}{\partial w}\frac{\partial w}{\partial y}, \tag{1}$$

where

$$R_u=\frac{2u}{u^2+v^2+w^2} \qquad R_v=\frac{2v}{u^2+v^2+w^2} \qquad R_w=\frac{2w}{u^2+v^2+w^2} \tag{2}$$

$$u_y=x^2e^{xy} \qquad v_y=\frac{xe^{xy}}{y}-\frac{e^{xy}}{y^2}=(xy-1)\frac{e^{xy}}{y^2} \qquad w_y=\frac{2x^2}{y^3\left(1+\frac{x^2}{y^2}\right)^2}=\frac{2x^2y}{\left(x^2+y^2\right)^2} \tag{3}$$

Substituting $u(0,1)=0$, $v(0,1)=1$, $w(0,1)=1$, and $x=0$, $y=1$ into (2) and (3) yields

$$R_u(0,1,1)=0 \qquad R_v(0,1,1)=1 \qquad R_w(0,1,1)=1 \tag{4}$$

$$u_y(0,1)=0 \qquad v_y(0,1)=-1 \qquad w_y(0,1)=1. \tag{5}$$

Finally, substituting (4) and (5) into (1) yields

$$\boxed{\frac{\partial R}{\partial y}=-1 \quad \text{at the point } x=0, y=1.}$$

3. (10 points) Find the equations for the tangent plane and normal line to the surface given by $z = \sin(x) + \sin(y) + \sin(2x + 3y)$ at the point $(0, 0, 0)$.

Solution: Generic equation of a plane is given by

$$\vec{n} \cdot (\vec{r} - \vec{r}_o) = 0, \tag{6}$$

where $\vec{r}_o$ is the position vector of a point in the plane and $\vec{n}$ is a normal vector to the plane. Let $f(x, y) = \sin(x) + \sin(y) + \sin(2x + 3y)$ and $F(x, y, z) = z - f(x, y) = 0$ then the normal to the plane is given by

$$\begin{aligned} \vec{n} &= \nabla F(x, y, z) \\ &= \langle -f_x, -f_y, 1 \rangle \\ &= \langle -\cos(x) - 2\cos(2x + 3y), -\cos(y) - 3\cos(2x + 3y), 1 \rangle . \end{aligned} \tag{7}$$

Substituting $x = 0$ and $y = 0$ into (7) yields

$$\vec{n} = \langle -3, -4, 1 \rangle . \tag{8}$$

Finally, substituting (8) and $\vec{r}_o = \langle 0, 0, 0 \rangle$ into (6) yields

$$\boxed{z = 3x + 4y \quad \text{equation of the tangent plane.}}$$

Generic equation of a line is given by

$$\vec{r} = \vec{r}_o + \vec{v}t,$$

where $\vec{r}_o$ is a position vector of a point on the line and $\vec{v}$ is a vector parallel to the line. In order to find the equation of the normal line to the surface, we set $\vec{v} = \vec{n}$ and $\vec{r}_o = \langle 0, 0, 0 \rangle$ to obtain

$$\boxed{\vec{r} = \langle -3t, -4t, t \rangle \text{ where } -\infty < t < \infty, \quad \text{equation of the normal line.}}$$

4. (15 points) Let $f(x,y,z) = z(x-y)^5 + xy^2z^3$.

(a) Find the directional derivative of f at $(2,1,-1)$ in the direction of $(1,1,1)$.

Solution: Directional derivative is given by

$$\mathrm{D}_{\hat{u}} f = \nabla f \cdot \hat{u}, \tag{9}$$

where $\hat{u}$ is a unit vector. We want to find the directional derivative at $(2,1,-1)$ <u>in the direction of</u> $(1,1,1)$, thus,

$$\begin{aligned} \vec{u} &= \langle 1,1,1\rangle - \langle 2,1,-1\rangle = \langle -1,0,2\rangle \\ \hat{u} &= \frac{\langle -1,0,2\rangle}{\sqrt{5}}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \nabla f(x,y,z) &= \left\langle 5z(x-y)^4 + y^2z^3, -5z(x-y)^4 + 2xyz^3, (x-y)^5 + 3xy^2z^2 \right\rangle \\ \nabla f(2,1,-1) &= \langle -6,1,7\rangle . \end{aligned} \tag{11}$$

Substitution (10) and (11) into (9) yields

$$\boxed{4\sqrt{5}.}$$

(b) In what direction is the maximum rate of change in f and what is the maximum rate of change of f?

Solution: The <u>direction</u> of the maximum rate of change of f is given by

$$\boxed{\alpha \langle -6,1,7\rangle , \text{ where } \alpha \text{ is any scalar greater than zero.}}$$

The maximum rate of change of f is given by

$$|\nabla f| = \sqrt{(-6)^2 + 1^2 + 7^2} = \boxed{\sqrt{86}.}$$

5. (12 points) Given $f(x,y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 1$. Find the critical points and classify each as a local maximum, local minimum or saddle point.

Solution: All first partial derivatives must vanish at the critical points, thus, we have

$$\begin{aligned} f_x &= 6xy - 6x = 0 \\ 0 &= x(y-1) \\ x &= 0 \quad \underline{\text{or}} \quad y = 1 \end{aligned}$$

$$\begin{aligned} f_y &= 3y^2 + 3x^2 - 6y = 0 \\ &\textbf{Case 1: } x = 0 \\ 0 &= y^2 - 2y \\ 0 &= y(y-2) \\ y &= 0 \quad \underline{\text{or}} \quad y = 2 \\ &\textbf{Case 2: } y = 1 \\ 1 &= x^2 \\ x &= 1 \quad \underline{\text{or}} \quad x = -1. \end{aligned}$$

Case 1 yields two critical points $(0,0),(0,2)$ and **Case 2** yields another two critical points $(1,1),(-1,1)$. Thus, we have total of four critical points, namely

$$\boxed{(0,0),(0,2),(1,1),(-1,1) \quad \text{critical points.}}$$

In order to classify the critical points we must compute the quantity

$$D = f_{xx}f_{yy} - f_{xy}^2, \tag{12}$$

where

$$f_{xx} = 6y - 6 \qquad f_{yy} = 6y - 6 \qquad f_{xy} = 6x, \tag{13}$$

for <u>each</u> critical point. Substituting (13) into (12) yields

$$D = 36\left[(y-1)^2 - x^2\right]. \tag{14}$$

It is clear from (14) that if $y = 1$ and $x \neq 0$ then $D < 0$, thus, critical points $\boxed{(1,1),(-1,1) \text{ are saddle points}}$. Substituting $x = 0$ and $y = 0$ into (14) yields $D(0,0) = 36 > 0$, and substituting $x = 0$ and $y = 0$ into (13) yields $f_{xx}(0,0) = -6 < 0$, thus, $\boxed{(0,0) \text{ is the local maximum point}}$. Substituting $x = 0$ and $y = 2$ into (14) yields $D(0,2) = 36 > 0$, and substituting $x = 0$ and $y = 2$ into (13) yields $f_{xx}(0,2) = 6 > 0$, thus, $\boxed{(0,2) \text{ is the local minimum point}}$.

6. (11 points) Calculate the integral $\int_0^1 \int_1^4 \frac{\sqrt{y}}{1+x^2}\, \mathrm{d}y\mathrm{d}x$.

Solution:

$$\begin{aligned}\int_0^1 \int_1^4 \frac{\sqrt{y}}{1+x^2}\, \mathrm{d}y\mathrm{d}x &= \left(\int_1^4 \sqrt{y}\, \mathrm{d}y\right)\left(\int_0^1 \frac{1}{1+x^2}\, \mathrm{d}x\right) \\ &= \left(\frac{2}{3}y^{3/2}\Big|_{y=1}^{y=4}\right)\left(\int_0^1 \frac{1}{1+x^2}\, \mathrm{d}x\right) \\ &= \frac{14}{3}\int_0^1 \frac{1}{1+x^2}\, \mathrm{d}x \qquad (15)\end{aligned}$$

In order to evaluate the integral in (15) consider the triangle in Figure 1.

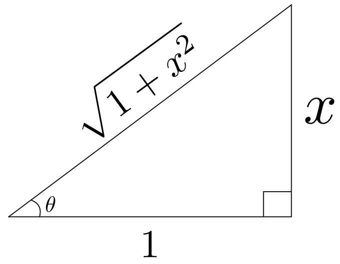

Figure 1: The triangle used in evaluation of (15).

From Figure 1, we have

$$x = \tan\theta \qquad 1 = \frac{1}{\cos^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} \qquad \mathrm{d}x = \frac{\mathrm{d}\theta}{\cos^2\theta} \qquad (16)$$

$$x = 0 \implies \theta = 0 \quad \text{and} \quad x = 1 \implies \theta = \frac{\pi}{4} \qquad (17)$$

$$\frac{1}{\sqrt{1+x^2}} = \cos\theta \qquad \frac{1}{1+x^2} = \cos^2\theta \qquad (18)$$

Substituting (16), (17) and (18) into (15) yields

$$\begin{aligned}\int_0^1 \int_1^4 \frac{\sqrt{y}}{1+x^2}\, \mathrm{d}y\mathrm{d}x &= \frac{14}{3}\int_0^{\pi/4} \cos^2\theta\, \frac{\mathrm{d}\theta}{\cos^2\theta} \\ &= \frac{14}{3}\int_0^{\pi/4} \mathrm{d}\theta \\ &= \frac{14}{3}\theta\Big|_{\theta=0}^{\theta=\pi/4} \\ &= \boxed{\frac{7\pi}{6}}.\end{aligned}$$

7. (12 points) Calculate the integral $\int_0^{\pi/4} \int_x^{\pi/4} \frac{\sin y}{y} \, \mathrm{d}y \mathrm{d}x$.

Solution: In order to calculate this integral we switch the limits of integration. The region of integration is bounded by $y = x, y = \pi/4, x = 0$ and $x = \pi/4$, see Figure 2.

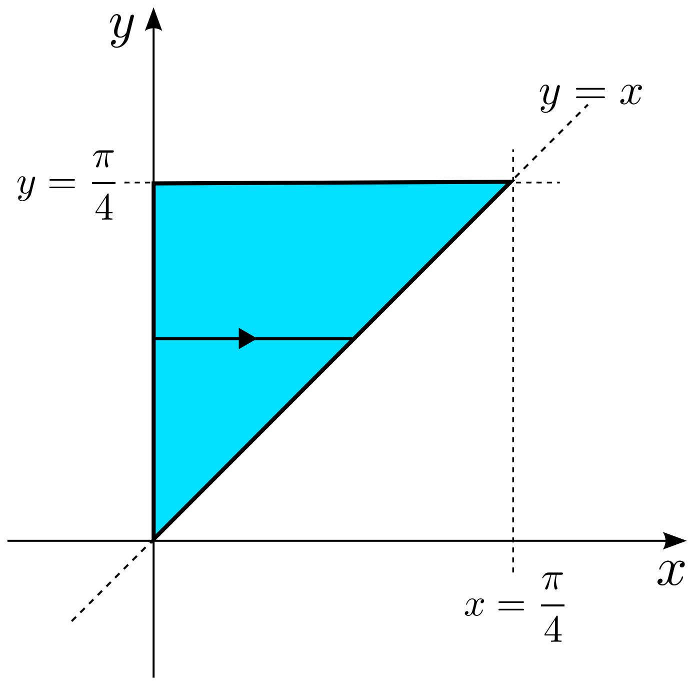

Figure 2: The region of integration is shown in light blue color.

Switching the limits of integration yields

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \int_x^{\pi/4} \frac{\sin y}{y} \, \mathrm{d}y \mathrm{d}x &= \int_0^{\frac{\pi}{4}} \int_0^{y} \frac{\sin y}{y} \, \mathrm{d}x \mathrm{d}y \\
&= \int_0^{\frac{\pi}{4}} \frac{\sin y}{y} x \Big|_{x=0}^{x=y} \, \mathrm{d}y \\
&= \int_0^{\frac{\pi}{4}} \sin y \, \mathrm{d}y \\
&= -\cos y \Big|_{y=0}^{y=\frac{\pi}{4}} \\
&= \boxed{\frac{\sqrt{2}-1}{\sqrt{2}}}.
\end{aligned}$$

8. (12 points) Evaluate the integral $\iint\limits_D e^{-x^2-y^2}\,\mathrm{d}A$ where D is the region bounded by the $x = \sqrt{4-y^2}$ and the y-axis.

Solution: The region of integration D is shown in Figure 3.

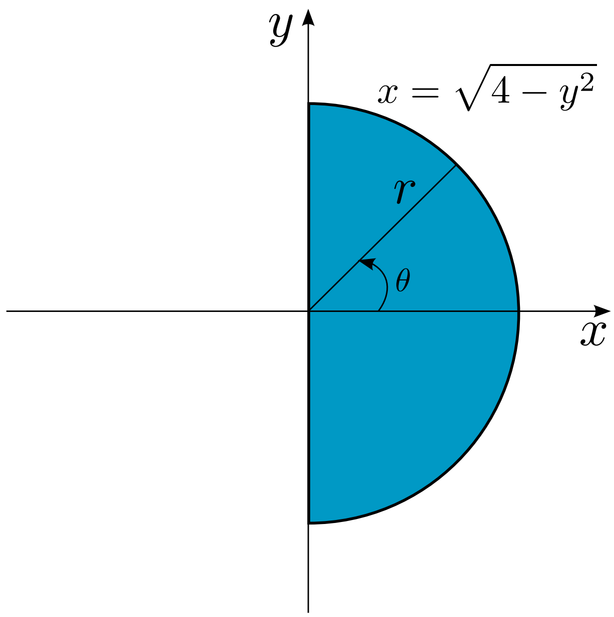

Figure 3: The region of integration is shown in light blue color. Recall that $x = \sqrt{4-y^2} \implies x^2+y^2 = 2^2$, thus, we have a half-disk of radius 2.

Converting the integral to polar coordinate system via $x = r\cos\theta, y = r\sin\theta$ and $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$ yields

$$\begin{aligned}\iint\limits_D e^{-x^2-y^2}\,\mathrm{d}A &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 e^{-r^2}\,r\mathrm{d}r\mathrm{d}\theta \\ &= -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-r^2}\Big|_{r=0}^{r=2}\,\mathrm{d}\theta \\ &= \boxed{\frac{\pi}{2}\left(1-e^{-4}\right).}\end{aligned}$$

9. (10 points) A tissue sample is modeled geometrically so that it occupies the region bounded by the line $y = x + 2$ and the parabola $y = x^2$. The density of the tissue is $\rho(x, y) = x^2$.

(a) **SET UP** the integrals to find the mass of the sample.

Solution: The region of integration is shown in Figure 4.

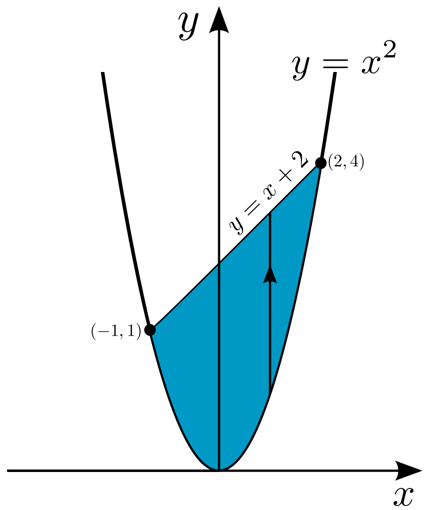

Figure 4: The region of integration is shown in light blue color. To find where the parabola and the line intersect, we set $x^2 = x + 2$ and solve for x to obtain $x = -1$, $x = 2$. Substituting $x = -1$ into $y = x^2$ yields $y = 1$ and substituting $x = 2$ into $y = x^2$ yields $y = 4$.

The mass of the tissue is given by

$$\begin{aligned} m &= \int_{-1}^{2} \int_{x^2}^{x+2} \rho(x, y)\, \mathrm{d}y \mathrm{d}x \\ &= \boxed{\int_{-1}^{2} \int_{x^2}^{x+2} x^2\, \mathrm{d}y \mathrm{d}x.} \end{aligned} \tag{19}$$

(b) **SET UP** the integrals for the x-component of the center of mass.

Solution: The x-component of the center of mass is given by

$$\begin{aligned} \bar{x} &= \frac{1}{m} \int_{-1}^{2} \int_{x^2}^{x+2} x\rho(x, y)\, \mathrm{d}y \mathrm{d}x \\ &= \boxed{\frac{1}{m} \int_{-1}^{2} \int_{x^2}^{x+2} x^3\, \mathrm{d}y \mathrm{d}x,} \end{aligned}$$

where m is given by (19).